

DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 13: NEWTON-RHAPSN APPROXIMATION

2006

7 (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation $x^3 + x - 9 = 0$, use the Newton-Raphson method to find x_2 , the second approximation.

SOLUTION

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots\dots \mathbf{16}$$

$$\text{For } n = 1: x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\text{For } n = 2: x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x) = x^3 + x - 9 \Rightarrow f'(x) = 3x^2 + 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 + 2 - 9}{3(2)^2 + 1} = 2 - \frac{1}{13} = \frac{25}{13}$$

STEPS

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

2005

7 (c) (i) Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

(ii) Hence use the Newton-Raphson method to show that the rule $u_{n+1} = \frac{(u_n)^2 + k}{2u_n}$

can be used to find increasingly accurate approximations for \sqrt{k} .

(iii) Using the above rule and taking $\frac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

SOLUTION

7 (c) (i)

Roots: $\sqrt{k}, -\sqrt{k}$

Quadratic Equation: $(x - \sqrt{k})(x + \sqrt{k}) = 0 \Rightarrow x^2 - k = 0$

7 (c) (ii)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots\dots \mathbf{16}$$

$$\text{For } n = 1: x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\text{For } n = 2: x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

STEPS

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^2 - k \Rightarrow f'(x) = 2x$$

$$f(u_n) = (u_n)^2 - k \text{ and } f'(u_n) = 2u_n$$

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)} = u_n - \frac{(u_n)^2 - k}{2u_n} = \frac{2(u_n)^2 - (u_n)^2 + k}{2u_n} = \frac{(u_n)^2 + k}{2u_n}$$

7 (c) (iii)

$$u_1 = \frac{3}{2}, k = 3$$

$$u_2 = \frac{(u_1)^2 + 3}{2u_1} = \frac{(\frac{3}{2})^2 + 3}{2(\frac{3}{2})} = \frac{\frac{9}{4} + 3}{3} = \frac{\frac{21}{4}}{3} = \frac{7}{4}$$

$$u_3 = \frac{(u_2)^2 + 3}{2u_2} = \frac{(\frac{7}{4})^2 + 3}{2(\frac{7}{4})} = \frac{\frac{49}{16} + 3}{\frac{7}{2}} \times \frac{16}{16} = \frac{49 + 48}{56} = \frac{97}{56}$$

2003

6 (b) Show that the equation $x^3 - 4x - 2 = 0$ has a root between 2 and 3.

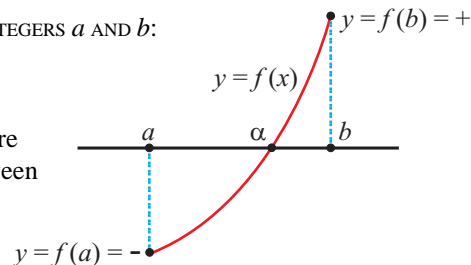
Taking $x_1 = 2$ as the first approximation to this root, use the Newton-Raphson method to find x_3 , the third approximation. Give your answer correct to two decimal places.

SOLUTION

TEST FOR A ROOT, α , BETWEEN INTEGERS a AND b :

1. Substitute in a : $f(a)$
2. Substitute in b : $f(b)$

If the sign of $f(a)$ and $f(b)$ are different, there is a root between a and b .



$$f(2) = (2)^3 - 4(2) - 2 = 8 - 8 - 2 = -2 < 0$$

$$f(3) = (3)^3 - 4(3) - 2 = 27 - 12 - 2 = 13 > 0$$

Therefore, there is a root between 2 and 3.

$$f(x) = x^3 - 4x - 2 \Rightarrow f'(x) = 3x^2 - 4$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{3(2)^2 - 4} = 2 - \frac{-2}{8} = 2.25$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.25 - \frac{f(2.25)}{f'(2.25)} = 2.25 - \frac{(2.25)^3 - 4(2.25) - 2}{3(2.25)^2 - 4} = 2.22$$

2001

7 (a) Taking $x_1 = 1$ as the first approximation to the real root of the equation $x^3 + x^2 - 1 = 0$, use the Newton-Rhapson method to find x_2 , the second approximation.

SOLUTION

7 (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots\dots \mathbf{16}$$

$$\text{For } n = 1: x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$\text{For } n = 2: x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

STEPS

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 + x^2 - 1 = 0 \Rightarrow f'(x) = 3x^2 + 2x$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^3 + 1^2 - 1}{3(1)^2 + 2(1)} = 1 - \frac{1}{5} = \frac{4}{5} = 0.8$$