DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 11: RATIONAL FUNCTIONS

2005

6 (c) The equation of a curve is $y=\dfrac{x}{x-1}$, where $x\neq 1$.

(i) Show that the curve has no local maximum or local minimum point.

(ii) Write down the equations of the asymptotes and hence sketch the curve.

(iii) Show that the curve is its own image under the symmetry in the point of intersection of the asymptotes.

SOLUTION

6 (c) (i)

Turning Point $\Rightarrow \dfrac{dy}{dx}=0$ .... **11** To find the turning points set $\dfrac{dy}{dx}=0$ and solve for x.

$$y=\frac{x}{x-1}\Rightarrow \frac{dy}{dx}=\frac{(x-1)1-x(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}$$

Turning points: $\dfrac{dy}{dx}=0\Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{(x-1)^2}=0\Rightarrow 1=0$ [There are no solutions and therefore, no local maximum or minimum point.]

6 (c) (ii)

FINDING THE VERTICAL ASYMPTOTE: Put the denominator equal to zero.

FINDING THE HORIZONTAL ASYMPTOTE: Find $\lim_{x\to\infty} y$.

HOW TO PLOT RATIONAL CURVES

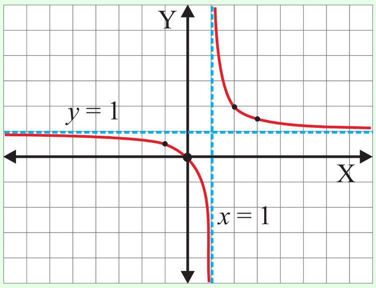
STEPS
1. Find the vertical and horizontal asymptotes.
2. Build up a table by choosing two points to the left and two points to the right of the vertical asymptote.
3. Plot the curves skimming along the asymptotes.

Vertical Asymptote: Put $x-1=0\Rightarrow x=1$

Horizontal Asymptote: $\lim_{x\to\infty} y=\lim_{x\to\infty}\dfrac{x}{x-1}=\lim_{x\to\infty}\dfrac{\not x}{\not x(1-\frac{1}{x})}=1$

Asymptotes: $x = 1$, $y = 1$

x	y	Point
-1	$\frac{1}{2}$	$(-1, \frac{1}{2})$
0	0	(0, 0)
1	Asymptote	
2	2	(2, 2)
3	$\frac{3}{2}$	$(3, \frac{3}{2})$

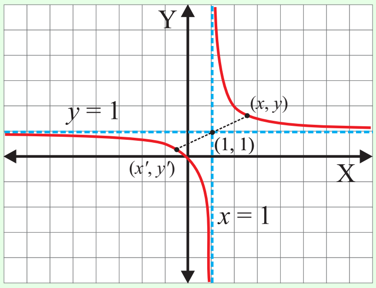


6 (c) (iii)

Choose any point (x, y) on the curve and translate it through (1, 1) to produce (x', y').

$(x, y) \to (1, 1) \to (2-x, 2-y)$

Replace x and y in the equation of the curve by $2-x$ and $2-y$.



$$y = \frac{x}{x-1} \Rightarrow 2-y = \frac{2-x}{2-x-1} \Rightarrow 2-y = \frac{2-x}{1-x}$$

$$\Rightarrow 2 - \frac{2-x}{1-x} = y \Rightarrow \frac{2(1-x)-1(2-x)}{1-x} = y$$

$$\Rightarrow \frac{2-2x-2+x}{1-x} = y \Rightarrow \frac{-x}{1-x} = y \Rightarrow \frac{x}{x-1} = y$$

2003

6 (c) The function $f(x) = \frac{1}{1-x}$ is defined for $x \in \mathbf{R} \setminus \{1\}$.

(i) Prove that the graph of f has no turning points and no points of inflection.

(ii) Write down the reason that justifies the statement: "f is increasing at every value of $x \in \mathbf{R} \setminus \{1\}$."

(iii) Given that $y = x + k$ is a tangent to the graph of f where k is a real number, find the two possible values of k.

SOLUTION

6 (c) (i)

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$\Rightarrow f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$\Rightarrow f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{(1-x)^2} = 0 \Rightarrow 1 = 0$$

Therefore, there are no turning points.

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{2}{(1-x)^3} = 0 \Rightarrow 2 = 0$$

Therefore, there are no points of inflection.

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ 11

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

Point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ 13

To find the point of inflection set $\frac{d^2y}{dx^2} = 0$ and solve for x.

6 (c) (ii)

$\frac{dy}{dx}=\frac{1}{(1-x)^2}>0$ for all values of x except $x = 1$. Therefore, f is increasing at every value of x except $x = 1$.

6 (c) (iii)

Tangent: $y = x + k$

Slope of tangent is 1.

$$\frac{dy}{dx}=\frac{1}{(1-x)^2}=1\Rightarrow 1=(1-x)^2\Rightarrow \pm 1=1-x\Rightarrow x=0,\ 2$$

$$y=f(x)=\frac{1}{1-x}$$

$x = 0$: $y=f(x)=\frac{1}{1-0}=1$ and $x = 2$: $y=f(x)=\frac{1}{1-2}=-1$

Therefore, a tangent of slope 1 exists on the function at (0, 1) and $(2, -1)$.

Hence, you can find two values of k.

Point (0, 1): $y=x+k\Rightarrow 1=0+k\Rightarrow k=1$

Point $(2, -1)$: $y=x+k\Rightarrow -1=2+k\Rightarrow k=-3$