Differentiation & Applications (Q 6 & 7, Paper 1)

Lesson No. 10: Cubic Functions

2006

6 (b) The equation of a curve is $y = 3x^4 - 2x^3 - 9x^2 + 8$.

(i) Show that the curve has a local maximum at the point (0, 8).

(ii) Find the coordinates of the two local minimum points on the curve.

(iii) Draw a sketch of the curve.

Solution

6 (b) (i)

Turning points: $\dfrac{dy}{dx} = 0 \Rightarrow 12x^3 - 6x^2 - 18x = 0 \Rightarrow x(2x^2 - x - 3) = 0$

$\Rightarrow x(2x-3)(x+1) = 0 \Rightarrow x = -1,\ 0,\ \frac{3}{2}$

At $x = 0 \Rightarrow y = 3(0)^4 - 2(0)^3 - 9(0)^2 + 8 = 8$

Therefore, (0, 8) is a turning point.

You now need to see whether it is a local maximum or minimum.

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 36(0)^2 - 12(0) - 18 = -18 < 0$$

Therefore, (0, 8) is a local maximum point.

6 (b) (ii)

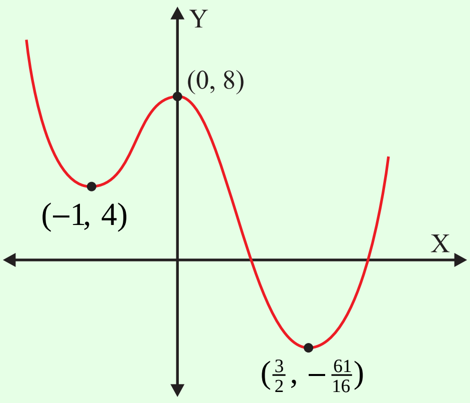
You have been told that the other two turning points are local minimums.

At $x = -1 \Rightarrow 3(-1)^4 - 2(-1)^3 - 9(-1)^2 + 8 = 3 + 2 - 9 + 8 = 4$

At $x = \frac{3}{2} \Rightarrow 3(\frac{3}{2})^4 - 2(\frac{3}{2})^3 - 9(\frac{3}{2})^2 + 8 = \frac{243}{16} - \frac{27}{4} - \frac{81}{4} + 8 = -\frac{61}{16}$

Therefore, $(-1, 4)$ and $(\frac{3}{2}, -\frac{61}{16})$ are the two local minimum points.

6 (b) (iii)



2004

6 (c) Let $f(x) = x^3 + 6x^2 + 15x + 36,\ x \in \mathbf{R}$.

(i) Show that $f'(x)$ can be written in the form $3[(x+a)^2 + b],\ a,\ b \in \mathbf{R}$, where $f'(x)$ is the first derivative of $f(x)$.

(ii) Hence show that $f(x) = 0$ has only one real root.

Solution

6 (c) (i)

$f(x) = x^3 + 6x^2 + 15x + 36 \Rightarrow f'(x) = 3x^2 + 12x + 15$

$\Rightarrow f'(x) = 3[x^2 + 4x + 5] = 3[x^2 + 4x + 4 + 1] = 3[(x+2)^2 + 1]$

6 (c) (ii)

$f'(x) = 3[(x+2)^2 + 1] > 0$ for all values of x. This means the graph of this function is always increasing. Therefore, it can only cut the X-axis once, i.e. one real root.

2002

6 (c) The function $f(x) = ax^3 + bx^2 + cx + d$ has a maximum point at (0, 4) and a point of inflection at (1, 0). Find the values of a, b, c and d.

Solution

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ 11

Point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ 13

$f(x) = ax^3 + bx^2 + cx + d$

(0, 4) and (1, 0) are on this curve.

$4 = a(0)^3 + b(0)^2 + c(0) + d \Rightarrow d = 4$

$0 = a(1)^3 + b(1)^2 + c(1) + 4 \Rightarrow a + b + c = -4 \ldots \mathbf{(1)}$

There is a turning point at (0, 4).

$$\left(\frac{dy}{dx}\right)_{x=0} = 3ax^2 + 2bx + c = 0 \Rightarrow 3a(0)^2 + 2b(0) + c = 0 \Rightarrow c = 0$$

There is a point of inflection at (1, 0).

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 6ax + 2b = 0 \Rightarrow 6a(1) + 2b = 0 \Rightarrow 3a + b = 0 \ldots \mathbf{(2)}$$

Now, equation (**1**) becomes $a + b = -4 \ldots \mathbf{(1)}$ as $c = 0$.

Solving equation (**1**) and (**2**) simultaneously $\Rightarrow a = 2,\ b = -6$

$\therefore f(x) = 2x^3 - 6x^2 + 4$

2001

7 (c) Let $g(x) = x^2 + \frac{a}{x^2}$ where a is a real number and $x \in \mathbf{R},\ x \neq 0$. Given that $g(x)$ has a turning point at $x = 2$,

(i) find the value of a

(ii) prove that $g(x)$ has no local maximum points.

SOLUTION

7 (c) (i)

$$g(x) = x^2 + \frac{a}{x^2} = x^2 + ax^{-2}$$

$$g'(x) = 2x - 2ax^{-3}$$

$$g''(x) = 2 + 6ax^{-4}$$

There is a turning point at x = 2:

$$\left(\frac{dy}{dx}\right)_{x=2} = 2x - 2ax^{-3} = 0 \Rightarrow 2(2) - \frac{2a}{2^3} = 0$$

$$\Rightarrow 4 - \frac{2a}{8} = 0 \Rightarrow 4 = \frac{2a}{8} \Rightarrow 32 = 2a \Rightarrow a = 16$$

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **11**

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} < 0$

Local Mininimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} > 0$ **12**

7 (c) (ii)

There are no local maxima if you can prove that $\frac{d^2y}{dx^2}$ is greater than zero for all values of x.

$$\frac{d^2y}{dx^2} = 2 + 12x^{-4} = 2 + \frac{12}{x^4} > 0$$ for all values of x.