2007

6 (a) Differentiate $\dfrac{x^2-1}{x^2+1}$ with respect to x.

(b) (i) Differentiate $\frac{1}{x}$ with respect to x from first principles.

(ii) Find the equation of the tangent to $y=\frac{1}{x}$ at the point $(2, \frac{1}{2})$.

(c) Let $f(x)=\tan^{-1}\frac{x}{2}$ and $g(x)=\tan^{-1}\frac{2}{x}$, for $x>0$.

(i) Find $f'(x)$ and $g'(x)$.

(ii) Hence, show that $f(x)$ and $g(x)$ is constant.

(iii) Find the value of $f(x)+g(x)$.

SOLUTION

6 (a)

THE QUOTIENT RULE: If $y=\dfrac{u}{v}$ then:

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} \quad ....\ \textbf{4}$$

$$y=\frac{x^2-1}{x^2+1}\Rightarrow\frac{dy}{dx}=\frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2}$$

$$=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

6 (b) (i)

FIRST PRINCIPLES PROOF 4. If $y=\dfrac{1}{x}\Rightarrow\dfrac{dy}{dx}=-\dfrac{1}{x^2}$.

PROOF

$$y+\Delta y=\frac{1}{x+\Delta x}$$

$$y=\frac{1}{x}$$

$$\Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-x-\Delta x}{x(x+\Delta x)}=-\frac{\Delta x}{x(x+\Delta x)} \quad \text{by subtraction}$$

$$\therefore\frac{\Delta y}{\Delta x}=-\frac{1}{x(x+\Delta x)}\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\frac{1}{x^2}$$

6 (b) (ii)

$$y=\frac{1}{x}\Rightarrow\frac{dy}{dx}=-\frac{1}{x^2}\Rightarrow\left(\frac{dy}{dx}\right)_{x=2}=-\frac{1}{4}=m$$

Eqn. of Tangent, T: $x+4y+k=0$

$(2,\tfrac{1}{2})\in T\Rightarrow 2+4(\tfrac{1}{2})+k=0\Rightarrow k=-4$

Eqn. of Tangent, T: $x+4y-4=0$

6 (c) (i)

INVERSE TAN

$$y=\tan^{-1}x\Rightarrow\frac{dy}{dx}=\frac{1}{1+x^2}$$ **10**

The formula in the tables is slightly different. In the tables let a = 1 to get formula **10**.

Formula 10 can be extended to:

$$y=\tan^{-1}f(x)\Rightarrow\frac{dy}{dx}=\frac{1}{1+f(x)^2}\times f'(x)$$ **10**

$$f(x)=\tan^{-1}(\tfrac{x}{2})\Rightarrow f'(x)=\frac{1}{1+(\frac{x}{2})^2}\times\tfrac{1}{2}=\frac{\frac{1}{2}}{(1+\frac{x^2}{4})}\times\frac{4}{4}=\frac{2}{x^2+4}$$

$$g(x)=\tan^{-1}(\tfrac{2}{x})\Rightarrow g'(x)=\frac{1}{1+(\frac{2}{x})^2}\times -2x^{-2}=\frac{-2}{x^2(1+\frac{4}{x^2})}=-\frac{2}{x^2+4}$$

6 (c) (ii)

$f'(x)+g'(x)=\dfrac{2}{x^2+4}-\dfrac{2}{x^2+4}=0\Rightarrow f(x)+g(x)=c,$ a constant. If you differentiate a constant, you get zero.

6 (c) (iii)

$f(x)+g(x)=c\Rightarrow\tan^{-1}(\tfrac{x}{2})+\tan^{-1}(\tfrac{2}{x})=c$

This is an identity. You can put any values you like in for x. Choose wisely. x = 2 is a good value to choose.

$\therefore\tan^{-1}(\tfrac{2}{2})+\tan^{-1}(\tfrac{2}{2})=\tan^{-1}1+\tan^{-1}1=\tfrac{\pi}{4}+\tfrac{\pi}{4}=\tfrac{\pi}{2}$

7 (a) Taking 1 as the first approximation of a root of $x^3+2x-4=0$, use the Newton-Raphson method to calculate the second approximation of this root.

(b) (i) Find the equation of the tangent to the curve $3x^2+y^2=28$ at the point $(2,-4)$.

(ii) $x=e^t\cos t$ and $y=e^t\sin t$. Show that $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$.

(c) $f(x)=\log_e 3x-3x$, where $x>0$.

(i) Show that $(\frac{1}{3},-1)$ is a local maximum point of $f(x)$.

(ii) Deduce that the graph of $f(x)$ does not intersect the x-axis.

SOLUTION

7 (a)

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad .... \quad \textbf{16}$$

For $n=1$: $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$.

For $n=2$: $x_3=x_2-\dfrac{f(x_2)}{f'(x_2)}$.

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

1. $f(x)=x^3+2x-4$

2. $f'(x)=3x^2+2$

3. $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}=1-\dfrac{f(1)}{f'(1)}=1-\dfrac{1+2-4}{3+2}=1-\dfrac{-1}{5}=1+\dfrac{1}{5}=\dfrac{6}{5}$

7 (b) (i)

$$3x^2+y^2=28\Rightarrow 6x+2y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{3x}{y}\Rightarrow\left(\frac{dy}{dx}\right)_{(2,-4)}=-\frac{3(2)}{(-4)}=\frac{3}{2}=m$$

Eqn. of T: $3x-2y+k=0$

$(2,-4)\in T\Rightarrow 3(2)-2(-4)+k=0\Rightarrow k=-14$

Eqn. of T: $3x-2y-14=0$

7 (b) (ii)

The Product Rule: If $y = u \times v$ then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$ **3**

$$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$ **5**

$$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$$ **6**

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$ **7**

Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$y = e^t \sin t \Rightarrow \frac{dy}{dx} = e^t(\cos t) + \sin t(e^t) = e^t(\cos t + \sin t)$$

$$x = e^t \cos t \Rightarrow \frac{dy}{dx} = e^t(-\sin t) + \cos t(e^t) = e^t(\cos t - \sin t)$$

$$\therefore \frac{dy}{dx} = \frac{e^t(\cos t + i\sin t)}{e^t(\cos t - i\sin t)} = \frac{(\cos t + i\sin t)}{(\cos t - i\sin t)}$$

$$\frac{x+y}{x-y} = \frac{e^t\cos t + e^t\sin t}{e^t\cos t - e^t\sin t} = \frac{e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)} = \frac{(\cos t + \sin t)}{(\cos t - \sin t)}$$

$$\therefore \frac{dy}{dx} = \frac{x+y}{x-y}$$

7 (c) (i)

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **11**

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$ **12**

$$f(x) = \ln 3x - 3x$$

$$\Rightarrow f'(x) = \tfrac{1}{x} - 3$$

$$\Rightarrow f''(x) = -\frac{1}{x^2}$$

Finding the turning point(s):

$$f'(x) = 0 \Rightarrow \tfrac{1}{x} - 3 = 0 \Rightarrow \tfrac{1}{x} = 3 \Rightarrow x = \tfrac{1}{3}$$

$$f(\tfrac{1}{3}) = \ln 3(\tfrac{1}{3}) - 3(\tfrac{1}{3}) = -1$$

Turning point: $(\tfrac{1}{3}, -1)$

There is only one solution and therefore, only one turning point.

$f''(x) = \left(\frac{d^2y}{dx^2}\right)_{(\frac{1}{3},-1)} = -\frac{1}{(\frac{1}{3})^2} = -9 < 0.$ This turning point is a local maximum.

7 (c) (ii)

The only turning point is a local maximum which is below the X-axis. There are no other turning points and so it is not possible for the graph of $f(x)$ to have any values above the local maximum. Therefore, $f(x)$ cannot cross the X-axis.