## 2006

6 (a) Differentiate $\sqrt{x}(x+2)$ with respect to $x$

(b) The equation of a curve is $y=3x^4-2x^3-9x^2+8$.

(i) Show that the curve has a local maximum at the point (0, 8).

(ii) Find the coordinates of the two local minimum points on the curve.

(iii) Draw a sketch of the curve.

(c) Prove by induction that $\frac{d}{dx}(x^n)=nx^{n-1},\ n\geq 1,\ n\in \mathbf{N}$.

**SOLUTION**

**6 (a)**

$y=\sqrt{x}(x+2)=x^{\frac{3}{2}}+2x^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}+x^{-\frac{1}{2}}=\frac{3}{2}\sqrt{x}+\frac{1}{\sqrt{x}}$

$y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}$ ....... **1**

6 (b)

Turning Point $\Rightarrow \frac{dy}{dx}=0$ ....... **11**

To find the turning points set $\frac{dy}{dx}=0$ and solve for $x$.

To find out if the turning point (TP) is a local maximum or local minimum:

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}<0$

Local Mininimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}>0$ ....... **12**

$y=3x^4-2x^3-9x^2+8$

$\Rightarrow \frac{dy}{dx}=12x^3-6x^2-18x$

$\Rightarrow \frac{d^2y}{dx^2}=36x^2-12x-18$

**6 (b) (i)**

Turning points: $\frac{dy}{dx}=0 \Rightarrow 12x^3-6x^2-18x=0 \Rightarrow x(2x^2-x-3)=0$

$\Rightarrow x(2x-3)(x+1)=0 \Rightarrow x=-1,\ 0,\ \frac{3}{2}$

At $x=0 \Rightarrow y=3(0)^4-2(0)^3-9(0)^2+8=8$

Therefore, (0, 8) is a turning point.

You now need to see whether it is a local maximum or minimum.

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 36(0)^2 - 12(0) - 18 = -18 < 0$$

Therefore, (0, 8) is a local maximum point.

**6 (b) (ii)**

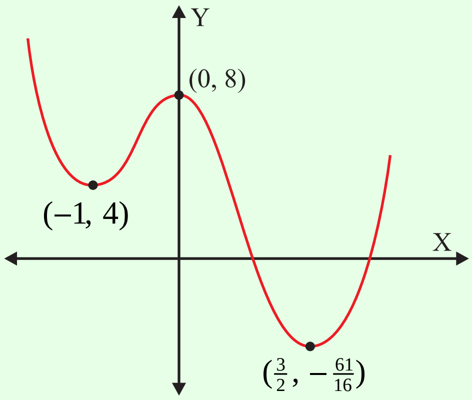
You have been told that the other two turning points are local minimums.

At $x = -1 \Rightarrow 3(-1)^4 - 2(-1)^3 - 9(-1)^2 + 8 = 3 + 2 - 9 + 8 = 4$

At $x = \frac{3}{2} \Rightarrow 3(\frac{3}{2})^4 - 2(\frac{3}{2})^3 - 9(\frac{3}{2})^2 + 8 = \frac{243}{16} - \frac{27}{4} - \frac{81}{4} + 8 = -\frac{61}{16}$

Therefore, $(-1, 4)$ and $(\frac{3}{2}, -\frac{61}{16})$ are the two local minimum points.

**6 (b) (iii)**




**6 (c)**

**Statement:** If $y = x^n$ prove $\frac{dy}{dx} = nx^{n-1}$ for all $n \in \mathbf{N}_0$.

**Proof**

**Step** 1. For $n = 1$: Prove $y = x^1 \Rightarrow \frac{dy}{dx} = 1$

$y = x$

$y + \Delta y = x + \Delta x$

$\Delta y = \Delta x$

$\therefore \frac{\Delta y}{\Delta x} = 1 \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 1$

**Step** 2. $n = k$: Assume $y = x^k \Rightarrow \frac{dy}{dx} = kx^{k-1}$

**Step** 3. $n = k + 1$: Prove $y = x^{k+1} \Rightarrow \frac{dy}{dx} = (k+1)x^k$

**Proof**: $y = x^{k+1} = x^k \times x \Rightarrow \frac{dy}{dx} = x^k \times 1 + x \times kx^{k-1}$ (Product Rule)

$= x^k + kx^k = x^k(k+1)$

7 (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation $x^3 + x - 9 = 0$, use the Newton-Raphson method to find $x_2$, the second approximation.

(b) The parametric equations of a curve are:

$$x = 3\cos\theta - \cos^3\theta$$
$$y = 3\sin\theta - \sin^3\theta, \text{ where } 0 < \theta < \tfrac{\pi}{2}.$$

(i) Find $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$.

(ii) Hence show that $\dfrac{dy}{dx} = \dfrac{-1}{\tan^3\theta}$.

(c) Given $y = \ln\left(\dfrac{3+x}{\sqrt{9-x^2}}\right)$, find $\dfrac{dy}{dx}$ and express it in the form $\dfrac{a}{b-x^n}$.

## Solution

**7 (a)**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{.......} \; \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

**Steps**
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value $x_n$ into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 + x - 9 \Rightarrow f'(x) = 3x^2 + 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 + 2 - 9}{3(2)^2 + 1} = 2 - \frac{1}{13} = \frac{25}{13}$$

**7 (b) (i)**

**Parametrics**: Do $\dfrac{dy}{dt}$ first, then do $\dfrac{dx}{dt}$, and then divide $\dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{dy}{dx}$

$$y = 3\sin\theta - \sin^3\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

$$\Rightarrow \frac{dy}{d\theta} = 3\cos\theta(1 - \sin^2\theta) = 3\cos\theta(\cos^2\theta) = 3\cos^3\theta$$

$$x = 3\cos\theta - \cos^3\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -3\sin\theta(1 - \cos^2\theta) = -3\sin\theta(\sin^2\theta) = -3\sin^3\theta$$

**7 (b) (ii)**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\frac{1}{\tan^3\theta}$$

**7 (c)**

$$y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$$ ....... **8**

REMEMBER IT AS:

One over the function inside the log × Differentiation of function

**STEPS**
1. Using log properties break up the log function.
2. Differentiate each log.
3. Tidy up the algebra at the end.

**1.** $y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right) = \ln(3+x) - \frac{1}{2}\ln(9-x^2)$ [Using log properties on page 36]

**2.** $\frac{dy}{dx} = \frac{1}{3+x} + \frac{2x}{2(9-x^2)} = \frac{1}{3+x} + \frac{x}{9-x^2}$

**3.** $\Rightarrow \frac{dy}{dx} = \frac{1}{(3+x)} + \frac{x}{(3+x)(3-x)} = \frac{1(3-x)+x}{(3+x)(3-x)} = \frac{3}{9-x^2}$