DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2005

6 (a) Differentiate with respect to x

(i) $(1+7x)^3$ (ii) $\sin^{-1}\left(\frac{x}{5}\right)$.

(b) Let $y=\frac{1-\cos x}{1+\cos x}$.

Show that $\frac{dy}{dx}=t+t^3$, where $t=\tan\left(\frac{x}{2}\right)$.

(c) The equation of a curve is $y=\frac{x}{x-1}$, where $x\neq 1$.

(i) Show that the curve has no local maximum or local minimum point.

(ii) Write down the equations of the asymptotes and hence sketch the curve.

(iii) Show that the curve is its own image under the symmetry in the point of intersection of the asymptotes.

SOLUTION

6 (a) (i)

$$y=[f(x)]^n \Rightarrow \frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)$$ 1

$$y=(1+7x)^3 \Rightarrow \frac{dy}{dx}=3(1+7x)^2(7)=21(1+7x)^2$$

6 (a) (ii)

$$y=\sin^{-1} f(x) \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-f(x)^2}}\times f'(x)$$ 9

$$y=\sin^{-1}\left(\frac{x}{5}\right)\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-(\frac{x}{5})^2}}\times\frac{1}{5}=\frac{1}{5\sqrt{1-\frac{x^2}{25}}}=\frac{1}{5\sqrt{\frac{25-x^2}{25}}}=\frac{1}{\sqrt{25-x^2}}$$

6 (b)

Use the quotient rule: $y=\frac{1-\cos x}{1+\cos x}$

$$\Rightarrow \frac{dy}{dx}=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\cos x\sin x}{(1+\cos x)^2}=\frac{2\sin x}{(1+\cos x)^2}$$

THE QUOTIENT RULE: If $y=\frac{u}{v}$ then:

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$ 4

The second part of this question involves proving a trig identity.

You are required to prove that $\frac{dy}{dx}=t+t^3 \Rightarrow \frac{2\sin x}{(1+\cos x)^2}=\tan\left(\frac{x}{2}\right)+\tan^3\left(\frac{x}{2}\right)$

Deal with half angles: Let $A=\frac{x}{2}\Rightarrow x=2A$

LHS

$\frac{2\sin 2A}{(1+\cos 2A)^2}=\frac{4\sin A\cos A}{(1+\cos^2 A-\sin^2 A)^2}$

$=\frac{4\sin A\cos A}{(2\cos^2 A)^2}=\frac{4\sin A\cos A}{4\cos^4 A}$

$=\frac{\sin A}{\cos^3 A}$

RHS

$\tan A+\tan^3 A=\tan A(1+\tan^2 A)$

$=\tan A(\sec^2 A)=\frac{\sin A}{\cos A}\times\frac{1}{\cos^2 A}$

$=\frac{\sin A}{\cos^3 A}$

6 (c) (i)

Turning Point $\Rightarrow \frac{dy}{dx}=0$ .... **11** To find the turning points set $\frac{dy}{dx}=0$ and solve for x.

$y=\frac{x}{x-1}\Rightarrow\frac{dy}{dx}=\frac{(x-1)1-x(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}$

Turning points: $\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{1}{(x-1)^2}=0\Rightarrow 1=0$ [There are no solutions and therefore, no local maximum or minimum point.]

6 (c) (ii)

Finding the vertical asymptote: Put the denominator equal to zero.

Finding the horizontal asymptote: Find $\lim_{x\to\infty} y$.

How to plot rational curves

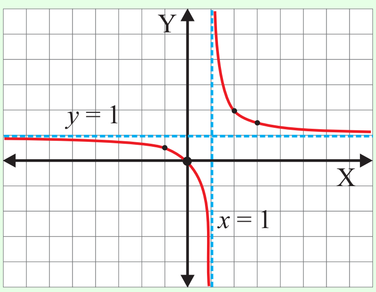
Steps
1. Find the vertical and horizontal asymptotes.
2. Build up a table by choosing two points to the left and two points to the right of the vertical asymptote.
3. Plot the curves skimming along the asymptotes.

Vertical Asymptote: Put $x-1=0\Rightarrow x=1$

Horizontal Asymptote: $\lim_{x\to\infty} y=\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}(1-\frac{1}{x})}=1$

Asymptotes: $x = 1$, $y = 1$

x	y	Point
-1	$\frac{1}{2}$	$(-1, \frac{1}{2})$
0	0	(0, 0)
1	Asymptote	
2	2	(2, 2)
3	$\frac{3}{2}$	$(3, \frac{3}{2})$

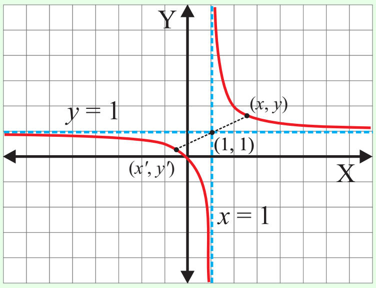


6 (c) (iii)

Choose any point (x, y) on the curve and translate it through (1, 1) to produce (x', y').

$(x, y) \to (1, 1) \to (2-x, 2-y)$

Replace x and y in the equation of the curve by $2-x$ and $2-y$.



$$y=\frac{x}{x-1} \Rightarrow 2-y=\frac{2-x}{2-x-1} \Rightarrow 2-y=\frac{2-x}{1-x}$$

$$\Rightarrow 2-\frac{2-x}{1-x}=y \Rightarrow \frac{2(1-x)-1(2-x)}{1-x}=y$$

$$\Rightarrow \frac{2-2x-2+x}{1-x}=y \Rightarrow \frac{-x}{1-x}=y \Rightarrow \frac{x}{x-1}=y$$

7 (a) Find from first principles the derivative of x^2 with respect to x.

(b) The parametric equations of a curve are:

$$x=8+\ln t^2$$
$$y=\ln(2+t^2), \text{ where } t>0.$$

(i) Find $\dfrac{dy}{dx}$ in terms of t and calculate its value at $t=\sqrt{2}$.

(ii) Find the slope of the tangent to the curve $xy^2+y=6$ at the point (1, 2).

(c) (i) Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

(ii) Hence use the Newton-Raphson method to show that the rule $u_{n+1}=\dfrac{(u_n)^2+k}{2u_n}$ can be used to find increasingly accurate approximations for $\sqrt{k}$.

(iii) Using the above rule and taking $\frac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

SOLUTION

7 (a)

$$y+\Delta y=(x+\Delta x)^2=x^2+2x\Delta x+(\Delta x)^2$$
$$y=x^2$$

$\Delta y=2x\Delta x+(\Delta x)^2$ by subtraction

$$\therefore \frac{\Delta y}{\Delta x}=2x+\Delta x \Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=2x$$

7 (b) (i)

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = 8 + \ln t^2 \Rightarrow \frac{dx}{dt} = \frac{2t}{t^2} = \frac{2}{t}$$

$$y = \ln(2 + t^2) \Rightarrow \frac{dy}{dt} = \frac{2t}{2 + t^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{2+t^2}}{\frac{2}{t}} = \frac{2t}{2+t^2} \times \frac{t}{2} = \frac{t^2}{2+t^2}$$

$$\left(\frac{dy}{dx}\right)_{t=\sqrt{2}} = \frac{(\sqrt{2})^2}{2 + (\sqrt{2})^2} = \frac{2}{2+2} = \frac{1}{2}$$

7 (b) (ii)

$$xy^2 + y = 6 \Rightarrow \left\{x(2y)\frac{dy}{dx} + y^2(1)\right\} + 1\frac{dy}{dx} = 0 \quad \text{[Notice the product]}$$

$$\Rightarrow 2xy\frac{dy}{dx} + y^2 + \frac{dy}{dx} = 0 \Rightarrow (2xy+1)\frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{(2xy+1)}$$

$$\left(\frac{dy}{dx}\right)_{(1,\,2)} = -\frac{2^2}{(2(1)(2)+1)} = -\frac{4}{4+1} = -\frac{4}{5}$$

7 (c) (i)

Roots: $\sqrt{k}, -\sqrt{k}$

Quadratic Equation: $(x - \sqrt{k})(x + \sqrt{k}) = 0 \Rightarrow x^2 - k = 0$

7 (c) (ii)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{.......} \quad \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^2 - k \Rightarrow f'(x) = 2x$$

$f(u_n) = (u_n)^2 - k$ and $f'(u_n) = 2u_n$

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)} = u_n - \frac{(u_n)^2 - k}{2u_n} = \frac{2(u_n)^2 - (u_n)^2 + k}{2u_n} = \frac{(u_n)^2 + k}{2u_n}$$

7 (c) (iii)

$u_1 = \frac{3}{2},\ k = 3$

$$u_2 = \frac{(u_1)^2 + 3}{2u_1} = \frac{(\frac{3}{2})^2 + 3}{2(\frac{3}{2})} = \frac{\frac{9}{4} + 3}{3} = \frac{\frac{21}{4}}{3} = \frac{7}{4}$$

$$u_3 = \frac{(u_2)^2 + 3}{2u_2} = \frac{(\frac{7}{4})^2 + 3}{2(\frac{7}{4})} = \frac{\frac{49}{16} + 3}{\frac{7}{2}} \times \frac{16}{16} = \frac{49 + 48}{56} = \frac{97}{56}$$