DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2004

6 (a) Differentiate $\frac{1}{2+5x}$ with respect to x.

(b) (i) Given $y=\tan^{-1}x$, find the value of $\frac{dy}{dx}$ at $x=\sqrt{2}$.

(ii) Differentiate, from first principles, $\cos x$ with respect to x.

(c) Let $f(x)=x^3+6x^2+15x+36,\ x\in\mathbf{R}$.

(i) Show that $f'(x)$ can be written in the form $3[(x+a)^2+b],\ a,\ b\in\mathbf{R}$, where $f'(x)$ is the first derivative of $f(x)$.

(ii) Hence show that $f(x)=0$ has only one real root.

SOLUTION

6 (a)

$$y=[f(x)]^n\Rightarrow\frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)$$ 1

$$y=\frac{1}{2+5x}=(2+5x)^{-1}\Rightarrow\frac{dy}{dx}=-1(2+5x)^{-2}(5)=-\frac{5}{(2+5x)^2}$$

6 (b) (i)

$$\left(\frac{dy}{dx}\right)_{x=\sqrt{2}}=\frac{1}{1+(\sqrt{2})^2}=\frac{1}{1+2}=\frac{1}{3}$$

$$y=\tan^{-1}x\Rightarrow\frac{dy}{dx}=\frac{1}{1+x^2}$$ 10

6 (b) (ii)

$$y=f(x)=\cos x$$
$$y+\Delta y=\cos(x+\Delta x)$$
$$y=\cos x$$

$$\therefore\ \Delta y=\cos(x+\Delta x)-\cos x=-2\sin(x+\tfrac{\Delta x}{2})\sin(\tfrac{\Delta x}{2})$$

$$\therefore\ \frac{\Delta y}{\Delta x}=-\sin(x+\tfrac{\Delta x}{2})\frac{\sin(\frac{\Delta x}{2})}{(\frac{\Delta x}{2})}$$

$$\therefore\ \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\sin x\times 1$$

$$=-\sin x$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$ 20

6 (c) (i)

$$f(x) = x^3 + 6x^2 + 15x + 36 \Rightarrow f'(x) = 3x^2 + 12x + 15$$

$$\Rightarrow f'(x) = 3[x^2 + 4x + 5] = 3[x^2 + 4x + 4 + 1] = 3[(x+2)^2 + 1]$$

6 (c) (ii)

$f'(x) = 3[(x+2)^2 + 1] > 0$ for all values of x. This means the graph of this function is always increasing. Therefore, it can only cut the X-axis once, i.e. one real root.

7 (a) An object's distance from a fixed point is given by $s = 12 + 24t - 3t^2$, where s is in metres and t is in seconds. Find the speed of the object when t = 3 seconds.

(b) The parametric equations of a curve are:

$$x = 2\theta - \sin 2\theta$$
$$y = 1 - \cos 2\theta, \text{ where } 0 < \theta < \pi.$$

(i) Find $\frac{dy}{dx}$.

(ii) Show that the tangent to the curve at $\theta = \frac{\pi}{6}$ is perpendicular to the tangent at $\theta = \frac{2\pi}{3}$.

(c) Given that $x = \dfrac{e^{2y} - 1}{e^{2y} + 1}$,

(i) show that $e^{2y} = \dfrac{1+x}{1-x}$

(ii) show that $\frac{dy}{dx}$ can be expressed in the form $\dfrac{p}{1-x^p}$, $p, q \in \mathbf{N}$.

Solution

7 (a)

$$s = 12 + 24t - 3t^2 \Rightarrow v = \frac{ds}{dt} = 24 - 6t$$

$v = \frac{ds}{dt}$ 14

$$\left(\frac{ds}{dt}\right)_{t=3} = 24 - 6(3) = 6\,\mathrm{m\,s^{-1}}$$

7 (b) (i)

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$y = 1 - \cos 2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin 2\theta$$

$$x = 2\theta - \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2 - 2\cos 2\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin 2\theta}{2 - 2\cos 2\theta} = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

7 (b) (ii)

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{6}} = \frac{\sin 2(\frac{\pi}{6})}{1 - \cos 2(\frac{\pi}{6})} = \frac{\sin(\frac{\pi}{3})}{1 - \cos(\frac{\pi}{3})} = \frac{\sin 60^{\circ}}{1 - \cos 60^{\circ}} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{2\pi}{3}} = \frac{\sin 2(\frac{2\pi}{3})}{1 - \cos 2(\frac{2\pi}{3})} = \frac{\sin(\frac{4\pi}{3})}{1 - \cos(\frac{4\pi}{3})} = \frac{\sin 240^{\circ}}{1 - \cos 240^{\circ}} = \frac{-\sin 60^{\circ}}{1 + \cos 60^{\circ}} = \frac{-\frac{\sqrt{3}}{2}}{1 + \frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = -\frac{1}{\sqrt{3}}$$

$\sqrt{3}\left(-\frac{1}{\sqrt{3}}\right) = -1$ [Therefore, the tangents are perpendicular.]

7 (c) (i)

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow x(e^{2y} + 1) = e^{2y} - 1 \Rightarrow xe^{2y} + x = e^{2y} - 1$$

$$\Rightarrow xe^{2y} - e^{2y} = -1 - x \Rightarrow e^{2y}(x - 1) = -1 - x \Rightarrow e^{2y} = \frac{-1 - x}{x - 1} = \frac{1 + x}{1 - x}$$

7 (c) (ii)

$$e^{2y} = \frac{1 + x}{1 - x} \Rightarrow 2e^{2y}\frac{dy}{dx} = \frac{(1 - x)(1) - (1 + x)(-1)}{(1 - x)^2}$$

$$\Rightarrow 2\left(\frac{1 + x}{1 - x}\right)\frac{dy}{dx} = \frac{1 - x + 1 - x}{(1 - x)^2} \Rightarrow 2\left(\frac{1 + x}{1 - x}\right)\frac{dy}{dx} = \frac{2}{(1 - x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1 - x)(1 + x)} = \frac{1}{1 - x^2}$$