Differentiation & Applications (Q 6 & 7, Paper 1)

2003

6 (a) Differentiate $\sqrt{1+4x}$ with respect to x.

(b) Show that the equation $x^3 - 4x - 2 = 0$ has a root between 2 and 3.
Taking $x_1 = 2$ as the first approximation to this root, use the Newton-Raphson method to find x_3, the third approximation. Give your answer correct to two decimal places.

(c) The function $f(x) = \dfrac{1}{1-x}$ is defined for $x \in \mathbf{R} \setminus \{1\}$.

(i) Prove that the graph of f has no turning points and no points of inflection.

(ii) Write down the reason that justifies the statement: "f is increasing at every value of $x \in \mathbf{R} \setminus \{1\}$."

(iii) Given that $y = x + k$ is a tangent to the graph of f where k is a real number, find the two possible values of k.

Solution

6 (a)

$$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$$ 1

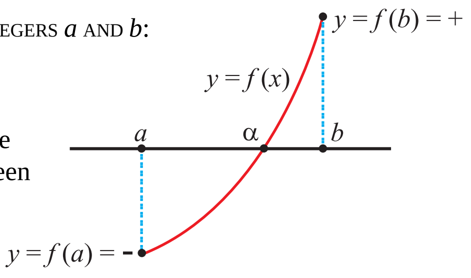
$$y = \sqrt{1+4x} = (1+4x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(1+4x)^{-\frac{1}{2}}(4) = \frac{2}{\sqrt{1+4x}}$$

6 (b)

Test for a root, α, between integers a and b:

1. Substitute in a: $f(a)$
2. Substitute in b: $f(b)$

If the sign of $f(a)$ and $f(b)$ are different, there is a root between a and b.



$$f(2) = (2)^3 - 4(2) - 2 = 8 - 8 - 2 = -2 < 0$$

$$f(3) = (3)^3 - 4(3) - 2 = 27 - 12 - 2 = 13 > 0$$

Therefore, there is a root between 2 and 3.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ 16

For $n = 1$: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

Steps

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 - 4x - 2 \Rightarrow f'(x) = 3x^2 - 4$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(x_1)} = 2 - \frac{-2}{3(2)^2 - 4} = 2 - \frac{-2}{8} = 2 \cdot 25$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2 \cdot 25 - \frac{f(2 \cdot 25)}{f'(2 \cdot 25)} = 2 \cdot 25 - \frac{(2 \cdot 25)^3 - 4(2 \cdot 25) - 2}{3(2 \cdot 25)^2 - 4} = 2 \cdot 22$$

6 (c) (i)

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$\Rightarrow f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$\Rightarrow f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{(1-x)^2} = 0 \Rightarrow 1 = 0$$

Therefore, there are no turning points.

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{2}{(1-x)^3} = 0 \Rightarrow 2 = 0$$

Therefore, there are no points of inflection.

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ 11

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

Point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ 13

To find the point of inflection set $\frac{d^2y}{dx^2} = 0$ and solve for x.

6 (c) (ii)

$\frac{dy}{dx} = \frac{1}{(1-x)^2} > 0$ for all values of x except $x = 1$. Therefore, f is increasing at every value of x except $x = 1$.

6 (c) (iii)

Tangent: $y = x + k$

Slope of tangent is 1.

$$\frac{dy}{dx} = \frac{1}{(1-x)^2} = 1 \Rightarrow 1 = (1-x)^2 \Rightarrow \pm 1 = 1 - x \Rightarrow x = 0,\ 2$$

$$y = f(x) = \frac{1}{1-x}$$

$x = 0$: $y = f(x) = \frac{1}{1-0} = 1$ and $x = 2$: $y = f(x) = \frac{1}{1-2} = -1$

Therefore, a tangent of slope 1 exists on the function at $(0, 1)$ and $(2, -1)$.

Hence, you can find two values of k.

Point $(0, 1)$: $y = x + k \Rightarrow 1 = 0 + k \Rightarrow k = 1$

Point $(2, -1)$: $y = x + k \Rightarrow -1 = 2 + k \Rightarrow k = -3$

7 (a) Differentiate each of the following with respect to x:

(i) $\cos^4 x$ (ii) $\sin^{-1}(\frac{x}{5})$.

(b) (i) The parametric equations of a curve are:

$x = \cos t + t\sin t$

$y = \sin t - t\cos t$ where $0 < t < \frac{\pi}{2}$.

Find $\frac{dy}{dx}$ and write your answer in its simplest form.

(ii) Given that $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$, find the value of $\frac{dy}{dx}$ at the point $(2, -3)$.

(c) (i) Given that $y = \ln\frac{1+x^2}{1-x^2}$ for $0 < x < 1$, find $\frac{dy}{dx}$ and write your answer in the form $\frac{kx}{1-x^k}$ where $k \in \mathbf{N}$.

(ii) Given that $f(\theta) = \sin(\theta + \pi)\cos(\theta - \pi)$, find the derivative of $f(\theta)$ and express it in the form $\cos n\theta$ where $n \in \mathbf{Z}$.

SOLUTION

7 (a) (i)

$$y = \cos^4 x = (\cos x)^4 \Rightarrow \frac{dy}{dx} = 4(\cos x)^3(-\sin x) = -4\cos^3 x \sin x$$

7 (a) (ii)

$$y = \sin^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-f(x)^2}} \times f'(x)$$ 9

$$y = \sin^{-1}\left(\frac{x}{5}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(\frac{x}{5})^2}} \times \frac{1}{5} = \frac{1}{5\sqrt{1-\frac{x^2}{25}}} = \frac{1}{5\sqrt{\frac{25-x^2}{25}}} = \frac{1}{\sqrt{25-x^2}}$$

7 (b) (i)

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$y = \sin t - t\cos t \Rightarrow \frac{dy}{dt} = \cos t - \{-t\sin t + \cos t\} = t\sin t$$

$$x = \cos t + t\sin t \Rightarrow \frac{dx}{dt} = -\sin t + \{t\cos t + \sin t\} = t\cos t$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\sin t}{t\cos t} = \tan t$$

7 (b) (ii)

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{6} \Rightarrow x^{-1}+y^{-1}=\frac{1}{6}$$

$$\Rightarrow -1x^{-2}-1y^{-2}\frac{dy}{dx}=0 \Rightarrow -x^{-2}=y^{-2}\frac{dy}{dx} \Rightarrow -\frac{x^{-2}}{y^{-2}}=\frac{dy}{dx} \Rightarrow -\frac{y^2}{x^2}=\frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)_{(2,\,-3)}=-\frac{(-3)^2}{2^2}=-\frac{9}{4}$$

7 (c) (i)

$$y=\ln f(x) \Rightarrow \frac{dy}{dx}=\frac{1}{f(x)}\times f'(x) \quad \ldots\ldots\ \textbf{8}$$

REMEMBER IT AS:

One over the function inside the log × Differentiation of function

STEPS
1. Using log properties break up the log function.
2. Differentiate each log.
3. Tidy up the algebra at the end.

1. $y=\ln\left(\frac{1+x^2}{1-x^2}\right) \Rightarrow y=\ln(1+x^2)-\ln(1-x^2)$

2. $\frac{dy}{dx}=\frac{2x}{(1+x^2)}+\frac{2x}{(1-x^2)}$

3. $\Rightarrow \frac{dy}{dx}=\frac{2x(1-x^2)+2x(1+x^2)}{(1+x^2)(1-x^2)}=\frac{2x-2x^3+2x+2x^3}{(1+x^2)(1-x^2)}=\frac{4x}{(1-x^4)}$

7 (c) (ii)

$$f(\theta)=\sin(\theta+\pi)\cos(\theta-\pi)=\sin(\theta+180^\circ)\cos(\theta-180^\circ)$$

$\sin(\theta+180^\circ)=-\sin\theta$ and $\cos(\theta-180^\circ)=-\cos\theta$ [You can work these out by expanding compound angles using the formulae on page 9 or by using ASTC and recognising them as well-behaved angles.]

$$\Rightarrow f(\theta)=(-\sin\theta)(-\cos\theta)=\sin\theta\cos\theta=\tfrac{1}{2}\sin 2\theta$$

$$\therefore f'(\theta)=\tfrac{1}{2}(2\cos 2\theta)=\cos 2\theta$$