DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2002

6 (a) Differentiate $(x^4+1)^5$ with respect to x.

(b) (i) Prove, from first principles, the addition rule:

if $f(x)=u(x)+v(x)$ then $\frac{df}{dx}=\frac{du}{dx}+\frac{dv}{dx}$.

(ii) Given $y=2x-\sin 2x$, find $\frac{dy}{dx}$. Give your answer in the form $k\sin^2 x$, where $k\in\mathbf{Z}$.

(c) The function $f(x)=ax^3+bx^2+cx+d$ has a maximum point at (0, 4) and a point of inflection at (1, 0). Find the values of a, b, c and d.

SOLUTION

6 (a)

$$\boxed{y=[f(x)]^n \Rightarrow \frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)} \quad \ldots\ldots\ 1$$

$$y=(x^4+1)^5 \Rightarrow \frac{dy}{dx}=5(x^4+1)^4\times 4x^3=20x^3(x^4+1)^4$$

6 (b) (i)

1. SUM RULE: If $y = u + v$ then $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

PROOF

$$\begin{aligned}
y+\Delta y &= (u+\Delta u)+(v+\Delta v)\\
y &= u+v\\
\hline
\Delta y &= \Delta u+\Delta v
\end{aligned}$$

$$\therefore \frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x} \Rightarrow \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

6 (b) (ii)

$$y=2x-\sin 2x \Rightarrow \frac{dy}{dx}=2-2\cos 2x=2(1-\cos 2x)$$

$$\Rightarrow \frac{dy}{dx}=2(1-\cos^2 x+\sin^2 x)=2(2\sin^2 x)=4\sin^2 x$$

6 (c)

$f(x) = ax^3 + bx^2 + cx + d$

(0, 4) and (1, 0) are on this curve.

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ 11

Point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ 13

$4 = a(0)^3 + b(0)^2 + c(0) + d \Rightarrow d = 4$

$0 = a(1)^3 + b(1)^2 + c(1) + 4 \Rightarrow a + b + c = -4...\mathbf{(1)}$

There is a turning point at (0, 4).

$$\left(\frac{dy}{dx}\right)_{x=0} = 3ax^2 + 2bx + c = 0 \Rightarrow 3a(0)^2 + 2b(0) + c = 0 \Rightarrow c = 0$$

There is a point of inflection at (1, 0).

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 6ax + 2b = 0 \Rightarrow 6a(1) + 2b = 0 \Rightarrow 3a + b = 0...\mathbf{(2)}$$

Now, equation (**1**) becomes $a + b = -4...\mathbf{(1)}$ as c = 0.

Solving equation (**1**) and (**2**) simultaneously $\Rightarrow a = 2,\ b = -6$

$\therefore\ f(x) = 2x^3 - 6x^2 + 4$

7 (a) Find the slope of the tangent to the curve $9x^2 + 4y^2 = 40$ at the point (2, 1).

(b) (i) Given that $y = \sin^{-1} 10x$, evaluate $\frac{dy}{dx}$ when $x = \frac{1}{20}$.

(ii) The parametric equations of a curve are $x = \ln(1 + t^2)$ and $y = \ln 2t$, where $t \in \mathbf{R},\ t > 0$. Find the value of $\frac{dy}{dx}$ when $t = \sqrt{5}$.

(c) Let $f(x) = \frac{e^x + e^{-x}}{2}$.

(i) Show that $f''(x) = f(x)$, where $f''(x)$ is the second derivative of $f(x)$.

(ii) Show that $\frac{f'(2x)}{f'(x)} = 2f(x)$ when $x \neq 0$ and where $f'(x)$ is the first derivative of $f(x)$.

Solution

7 (a)

$$9x^2 + 4y^2 = 40 \Rightarrow 18x + 8y\frac{dy}{dx} = 0 \Rightarrow 8y\frac{dy}{dx} = -18x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{18x}{8y} = -\frac{9x}{4y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,\,1)} = -\frac{9(2)}{4(1)} = -\frac{18}{4} = -\frac{9}{2}$$

7 (b) (i)

$$\boxed{y=\sin^{-1} f(x) \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-f(x)^2}}\times f'(x)} \;\; \ldots\ldots\; 9$$

$$y=\sin^{-1}10x \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-(10x)^2}}\times 10=\frac{10}{\sqrt{1-100x^2}}$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{1}{20}}=\frac{10}{\sqrt{1-100(\frac{1}{20})^2}}=\frac{10}{\sqrt{1-\frac{1}{4}}}=\frac{10}{\sqrt{\frac{3}{4}}}=\frac{20}{\sqrt{3}}$$

7 (b) (ii)

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{dy}{dx}$

$$y=\ln 2t \Rightarrow \frac{dy}{dt}=\frac{2}{2t}=\frac{1}{t}$$

$$x=\ln(1+t^2) \Rightarrow \frac{dx}{dt}=\frac{2t}{(1+t^2)}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{t}}{\frac{2t}{(1+t^2)}}=\frac{1}{t}\times\frac{(1+t^2)}{2t}=\frac{(1+t^2)}{2t^2}$$

$$\left(\frac{dy}{dx}\right)_{t=\sqrt{5}}=\frac{(1+(\sqrt{5})^2)}{2(\sqrt{5})^2}=\frac{1+5}{10}=\frac{3}{5}$$

7 (c) (i)

$$\boxed{y=e^x \Rightarrow \frac{dy}{dx}=e^x} \;\; \ldots\ldots\; 7$$

Formula 7 can be extended to:

$$\boxed{y=e^{f(x)} \Rightarrow \frac{dy}{dx}=e^{f(x)}\times f'(x)} \;\; \ldots\ldots\; 7$$

REMEMBER IT AS:

Repeat the whole function × Differentiation of the power.

$$f(x)=\tfrac{1}{2}(e^x+e^{-x})$$

$$\Rightarrow f'(x)=\tfrac{1}{2}(e^x-e^{-x})$$

$$\Rightarrow f''(x)=\tfrac{1}{2}(e^x+e^{-x})$$

$$\Rightarrow f(x)=f''(x)$$

7 (c) (ii)

Show $\dfrac{f'(2x)}{f'(x)}=2f(x)$

LHS

$$\frac{f'(2x)}{f'(x)}=\frac{\frac{1}{2}(e^{2x}-e^{-2x})}{\frac{1}{2}(e^x-e^{-x})}$$

$$=\frac{(e^x-e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})}$$

$$=(e^x+e^{-x})$$

RHS

$$2f(x)=2\times\tfrac{1}{2}(e^x+e^{-x})$$

$$=(e^x+e^{-x})$$