DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2001

6 (a) Differentiate $\frac{x}{1+x^2}$ with respect to x.

(b) (i) Given that $y=\sqrt{x}$, what is $\frac{dy}{dx}$?

(ii) Now, find from first principles the derivative of $\sqrt{x}$ with respect to x.

(c) Let $x=t^2e^t$ and $y=t+2\ln t$ for $t>0$.

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(ii) Hence, show that $\frac{dy}{dx}=\frac{1}{x}$.

SOLUTION

6 (a)

THE QUOTIENT RULE: If $y=\frac{u}{v}$ then: $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ 4

$$y=\frac{x}{1+x^2}\Rightarrow\frac{dy}{dx}=\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\frac{1+x^2-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

6 (b) (i)

$$y=\sqrt{x}=x^{\frac{1}{2}}\Rightarrow\frac{dy}{dx}=\tfrac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$

$y=x^n\Rightarrow\frac{dy}{dx}=nx^{n-1}$ 1

6 (b) (ii)

$$y=f(x)=\sqrt{x}$$
$$y+\Delta y=\sqrt{x+\Delta x}$$
$$y=\sqrt{x}$$

$$\therefore \Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\frac{\sqrt{x+\Delta x}-\sqrt{x}}{1}\times\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$\therefore \Delta y=\frac{x+\Delta x-x}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{\Delta x}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$\therefore \frac{\Delta y}{\Delta x}=\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{2\sqrt{x}}$$

6 (c) (i)

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = t^2e^t \Rightarrow \frac{dx}{dt} = t^2e^t + e^t(2t) = te^t(t+2)$$

$$y = t + 2\ln t \Rightarrow \frac{dy}{dt} = 1 + \frac{2}{t}$$

6 (c) (ii)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\frac{2}{t}}{te^t(t+2)} \times \frac{t}{t} = \frac{t+2}{t^2e^t(t+2)} = \frac{1}{t^2e^t} = \frac{1}{x}$$

7 (a) Taking $x_1 = 1$ as the first approximation to the real root of the equation $x^3 + x^2 - 1 = 0$, use the Newton-Rhapson method to find x_2, the second approximation.

(b) (i) Differentiate $\tan^{-1} 7x$ with respect to x.

(ii) Given that $y = \sin x \cos x$, find $\frac{dy}{dx}$ and express it in the form $\cos nx$ where $n \in \mathbf{Z}$.

(c) Let $g(x) = x^2 + \frac{a}{x^2}$ where a is a real number and $x \in \mathbf{R}$, $x \neq 0$. Given that $g(x)$ has a turning point at $x = 2$,

(i) find the value of a

(ii) prove that $g(x)$ has no local maximum points.

SOLUTION

7 (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots\ \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 + x^2 - 1 = 0 \Rightarrow f'(x) = 3x^2 + 2x$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^3 + 1^2 - 1}{3(1)^2 + 2(1)} = 1 - \frac{1}{5} = \frac{4}{5} = 0 \cdot 8$$

7 (b) (i)

$$y = \tan^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{1 + f(x)^2} \times f'(x)$$ 10

$$y = \tan^{-1} 7x \Rightarrow \frac{dy}{dx} = \frac{1}{1+(7x)^2} \times 7 = \frac{7}{1+49x^2}$$

7 (b) (i)

$$y = \sin x \cos x = \tfrac{1}{2}\sin 2x$$

$$\sin 2A = 2\sin A \cos A$$ 13

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(2\cos 2x) = \cos 2x$$

7 (c) (i)

$$g(x) = x^2 + \frac{a}{x^2} = x^2 + ax^{-2}$$

$$g'(x) = 2x - 2ax^{-3}$$

$$g''(x) = 2 + 6ax^{-4}$$

There is a turning point at x = 2:

$$\left(\frac{dy}{dx}\right)_{x=2} = 2x - 2ax^{-3} = 0 \Rightarrow 2(2) - \frac{2a}{2^3} = 0$$

$$\Rightarrow 4 - \frac{2a}{8} = 0 \Rightarrow 4 = \frac{2a}{8} \Rightarrow 32 = 2a \Rightarrow a = 16$$

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ 11

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Mininimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$ 12

7 (c) (ii)

There are no local maxima if you can prove that $\frac{d^2y}{dx^2}$ is greater than zero for all values of x.

$$\frac{d^2y}{dx^2} = 2 + 12x^{-4} = 2 + \frac{12}{x^4} > 0$$ for all values of x.