COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

SOLUTIONS NO. 5: MATRIX EQUATIONS

2006

3 (b) (i) Use matrix methods to solve the simultaneous equations

$$4x-2y=5$$
$$8x+3y=-4$$

(ii) Find the two values of k which satisfy the matrix equation

$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3 & 4\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11$$

SOLUTION

3 (b) (i)

Simultaneous equations in 2 or more unknowns can be written as a single matrix equation:

$$AX=B\Rightarrow A^{-1}AX=A^{-1}B$$

The simultaneous equations can be written in matrix form as follows:

$$\begin{pmatrix}4 & -2\\ 8 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}5\\ -4\end{pmatrix}\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{28}\begin{pmatrix}3 & 2\\ -8 & 4\end{pmatrix}\begin{pmatrix}5\\ -4\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{28}\begin{pmatrix}7\\ -56\end{pmatrix}\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\ -2\end{pmatrix}$$

ANSWER: $x=\frac{1}{4},\ y=-2$

3 (b)

$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3 & 4\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11\Rightarrow\begin{pmatrix}3-2k & k+4\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11$$

$$\Rightarrow 3-2k+k^2+4k=11\Rightarrow k^2+2k-8=0$$
$$\Rightarrow (k+4)(k-2)=0\Rightarrow k=-4,\ 2$$

2004

3 (c) Let $A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$ and $P=\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}$.

(i) Evaluate $A^{-1}PA$ and hence $(A^{-1}PA)^{10}$.

(ii) Use the fact that $(A^{-1}PA)^{10}=A^{-1}P^{10}A$ to evaluate P^{10}.

SOLUTION

3 (c) (i)

$A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$ and $P=\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}$.

$$D=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\Rightarrow D^n=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^n=\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix}$$ **7**

Evaluate $A^{-1}PA$ and hence $(A^{-1}PA)^{10}$.

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ **8**

$$A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{2-3}\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}=\frac{1}{-1}\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}$$

$$A^{-1}PA=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

The result is a diagonal matrix which means you can use formula 7.

$$(A^{-1}PA)^{10}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}^{10}=\begin{pmatrix}1^{10} & 0\\ 0 & 2^{10}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}$$

3 (c) (ii)

You are told that $(A^{-1}PA)^{10}=A^{-1}P^{10}A$ and are asked to find P^{10}.

$$\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}=A^{-1}P^{10}A\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}P^{10}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$$

To isolate P^{10}, multiply each side by matrix A at the front and A^{-1} at the end.

Therefore, $(A^{-1}PA)^{10}=A^{-1}P^{10}A\Rightarrow A(A^{-1}PA)^{10}A^{-1}=AA^{-1}P^{10}AA^{-1}$

$$\Rightarrow A(A^{-1}PA)^{10}A^{-1}=P^{10}$$

$$\Rightarrow P^{10}=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -3072\\ -1 & 2048\end{pmatrix}\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}=\begin{pmatrix}3070 & 3069\\ -2046 & -2045\end{pmatrix}$$

2001

3 (b) (i) Write the simultaneous equations

$$x-\sqrt{3}y=-2$$
$$\sqrt{3}x+y=2\sqrt{3}$$

in the form $A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix}$ where A is a 2×2 matrix.

(ii) Then, find A^{-1} and use it to solve the equations for *x* and *y*.

SOLUTION

3 (b) (i)

$$\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix}$$

3 (b) (ii)

$$A=\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{1+3}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}4\\4\sqrt{3}\end{pmatrix}=\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$$

$\Rightarrow x=1$ and $y=\sqrt{3}$.