COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

SOLUTIONS NO. 4: MATRIX ALGEBRA

2005

3 (a) Given that $A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, show that $A^3=A^{-1}$.

SOLUTION

A is a diagonal matrix. Therefore, the following property holds:

$$D=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\Rightarrow D^n=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^n=\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix} \quad \text{....... } 7$$

The inverse of matrix *A* is given by:

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} \quad \text{....... } 8$$

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Rightarrow A^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^3=\begin{pmatrix}1^3 & 0\\ 0 & (-1)^3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

2003

3 (a) Evaluate $\begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}3 & 0\\ -5 & 1\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix}$.

SOLUTION

$$\begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}3 & 0\\ -5 & 1\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix}=\begin{pmatrix}13 & -2\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix}=(17)$$

2002

3 (c) The following three statements are true whenever x and y are real numbers:

- $x + y = y + x$
- $xy = yx$
- If $xy = 0$ then either $x = 0$ or $y = 0$.

Investigate whether the statements are also true when x is the matrix $\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$ and y is the matrix $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$.

Solution

- $x + y = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix}$ and $y + x = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix}$

  $\therefore x + y = y + x$ (True)

- $xy = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $yx = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} -12 & 4 \\ -36 & 12 \end{pmatrix}$

  $\therefore xy \neq yx$ (False)

- If $xy = 0$ then either $x = 0$ or $y = 0$. This is false as it was already shown in the previous part $xy = 0$ even though $x \neq 0$ and $y \neq 0$.

2001

3 (c) (i) Write $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $ax^2 + bxy + cy^2$ where $a, b, c \in \mathbf{Z}$.

Solution

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x - 4y & 3x + 5y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-2x^2 - 4xy + 3xy + 5y^2)$$

$$= (-2x^2 - xy + 5y^2)$$