Complex Numbers & Matrices (Q 3, Paper 1)

Solutions No. 3: De Moivre's Theorem

2006

3 (c) (i) Express $-8-8\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$.

(ii) Hence find $(-8-8\sqrt{3}i)^3$.

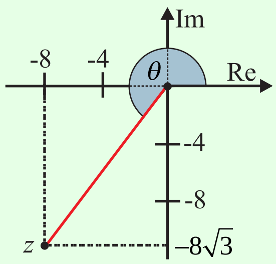
(iii) Find the four complex number z such that $z^4 = -8-8\sqrt{3}i$. Give your answers in the form $a+bi$, with a and b fully evaluated.

Solution

3 (c) (i)

Steps

1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z = r(\cos\theta + i\sin\theta)$.



1. $r = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{64 + 3\times 64} = 16$

2. Draw picture.

3. $|\tan\theta| = \left|\dfrac{-8\sqrt{3}}{-8}\right| = \sqrt{3} \Rightarrow \theta = 60^\circ$

Angle is in third quadrant $\Rightarrow \theta = 240^\circ = 240^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{4\pi}{3}$

4. $z = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

3 (c) (ii)

Use De Moivre's Theorem: $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **4**

$z^3 = (-8-8\sqrt{3})^3 = [16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})]^3$

$= 16^3(\cos 4\pi + i\sin 4\pi) = 4096(1+0i) = 4096$

3 (c) (iii)

To find roots, write z in general polar form.

$$z = 16\{\cos(\tfrac{4\pi}{3} + 2n\pi) + i\sin(\tfrac{4\pi}{3} + 2n\pi)\} = 16\left\{\cos\left(\frac{4\pi + 6n\pi}{3}\right) + i\sin\left(\frac{4\pi+6n\pi}{3}\right)\right\}$$

$$z^{\frac{1}{4}} = 16^{\frac{1}{4}}\left\{\cos\left(\frac{4\pi + 6n\pi}{3}\right) + i\sin\left(\frac{4\pi+6n\pi}{3}\right)\right\}^{\frac{1}{4}} = 2\left\{\cos\left(\frac{4\pi + 6n\pi}{12}\right) + i\sin\left(\frac{4\pi+6n\pi}{12}\right)\right\}$$

$$\Rightarrow z^{\frac{1}{4}} = 2\left\{\cos\left(\frac{2\pi + 3n\pi}{6}\right) + i\sin\left(\frac{2\pi+3n\pi}{6}\right)\right\}$$

- $n=0 \Rightarrow z_1 = 2\left\{\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right\} = 2\{\cos 60^\circ + i\sin 60^\circ\} = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 1+\sqrt{3}i$

- $n=1 \Rightarrow z_2 = 2\left\{\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right\} = 2\{\cos 150^\circ + i\sin 150^\circ\}$

  $= 2\{-\cos 30^\circ + i\sin 30^\circ\} = 2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = -\sqrt{3}+i$

- $n=2 \Rightarrow z_3 = 2\left\{\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right\} = 2\{\cos 240^\circ + i\sin 240^\circ\}$

  $= 2\{-\cos 60^\circ - i\sin 60^\circ\} = 2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = -1-\sqrt{3}i$

- $n=3 \Rightarrow z_3 = 2\left\{\cos\left(\frac{11\pi}{6}\right)+i\sin\left(\frac{11\pi}{6}\right)\right\} = 2\{\cos 330^\circ + i\sin 330^\circ\}$

  $= 2\{\cos 30^\circ - i\sin 30^\circ\} = 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = \sqrt{3}-i$

Answers: $1+\sqrt{3}i,\ -\sqrt{3}+i,\ -1-\sqrt{3}i,\ \sqrt{3}-i$

2005

3 (c) (i) $z=\cos\theta+i\sin\theta$. Use De Moivre's theorem to show that $z^n+\frac{1}{z^n}=2\cos n\theta$, for $n\in\mathbf{N}$.

(ii) Expand $\left(z+\frac{1}{z}\right)^4$ and hence express $\cos^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$.

SOLUTION

3 (c) (i)

$z^n+\frac{1}{z^n}=z^n+z^{-n}=(\cos\theta+i\sin\theta)^n+(\cos\theta+i\sin\theta)^{-n}$

$=(\cos n\theta+i\sin n\theta)+(\cos(-n\theta)+i\sin(-n\theta))=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$=2\cos n\theta$

3 (c) (ii)

$$\left(z+\frac{1}{z}\right)^4 = z^4+4z^3\left(\frac{1}{z}\right)+6z^2\left(\frac{1}{z}\right)^2+4z^3\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4$$

$$\Rightarrow\left(z+\frac{1}{z}\right)^4 = z^4+4z^2+6+4\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^4}\right)$$

$$\Rightarrow\left(z+\frac{1}{z}\right)^4=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$

Using the result in part (i):

$$\Rightarrow(2\cos\theta)^4=(2\cos4\theta)+4(2\cos2\theta)+6$$

$$\Rightarrow16\cos^4\theta=2\cos4\theta+8\cos2\theta+6$$

$$\Rightarrow\cos^4\theta=\tfrac{1}{8}(\cos4\theta+4\cos2\theta+3)$$

2004

3 (b) (i) $z_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ and $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$. Evaluate z_1z_2, giving your answer in the form $x+iy$.

SOLUTION

RULES FOR COMBINING OBJECTS IN POLAR FORM

1. $(\cos A\oplus i\sin A)(\cos B\oplus i\sin B)=\cos(A+B)+i\sin(A+B)$

$z_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ and $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$.

$$\Rightarrow z_1z_2=\cos\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)=\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$$

$$=\cos300^\circ+i\sin300^\circ=\cos60^\circ-i\sin60^\circ=\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i$$

2003

3 (c) $1,\ \omega,\ \omega^2$ are the three roots of the equation $z^3-1=0$.

(i) Prove that $1+\omega+\omega^2=0$.

(ii) Hence, find the value of $(1-\omega-\omega^2)^5$.

SOLUTION

3 (c) (i)

$$z^3=1\Rightarrow z=(1+0i)^{\frac{1}{3}}=(\cos2n\pi+i\sin2n\pi)^{\frac{1}{3}}=\cos\left(\frac{2n\pi}{3}\right)+i\sin\left(\frac{2n\pi}{3}\right)$$

$$n=0\Rightarrow z_1=\cos0^\circ+i\sin0^\circ=1$$

$$n=1\Rightarrow z_2=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=\cos120^\circ+i\sin120^\circ=-\cos60^\circ+i\sin60^\circ=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i$$

$$n=2\Rightarrow z_3=\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)=\cos240^\circ+i\sin240^\circ=-\cos60^\circ-i\sin60^\circ=-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i$$

These 3 roots are called $1,\ \omega,\ \omega^2$.

$$\therefore1+\omega+\omega^2=1-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i=0$$

3 (c) (ii)

$1+\omega+\omega^2=0 \Rightarrow 1=-\omega-\omega^2$

$\therefore (1-\omega-\omega^2)^5=2^5=32$

2002

3 (a) Express $-1+\sqrt{3}i$ in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$.

SOLUTION

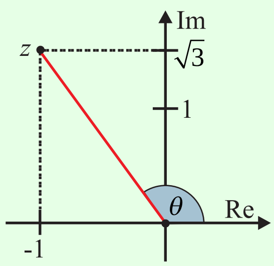
3 (a)

STEPS

1. Find $r=|z|=\sqrt{\text{Re}^2+\text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta|=\left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z=r(\cos\theta+i\sin\theta)$.

1. $z=-1+\sqrt{3}i \Rightarrow r=|z|=\sqrt{(-1)^2+(\sqrt{3})^2}=2$

2. Draw a picture.



3. $\tan\theta=\left|\dfrac{\sqrt{3}}{-1}\right|=\sqrt{3}\Rightarrow\theta=60^\circ$ (First quadrant)

$\therefore \theta=120^\circ=\dfrac{2\pi}{3}$ (Second quadrant)

4. $\therefore z=2(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$