Complex Numbers & Matrices (Q 3, Paper 1)

2011

3. (a) Express $\frac{1+2i}{2-i}$ in the form of $a+bi$, where $i^2=-1$.

(b) (i) Find the two complex numbers such that

$$(a+bi)^2=-3+4i.$$

(ii) Hence solve the equation

$$x^2+x+1-i=0.$$

(c) (i) Let A and B be 2×2 matrices, where A has an inverse.

Show that $(A^{-1}BA)^n=A^{-1}B^nA$ for all $n\in\mathbb{N}$.

Let $P=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$ and $M=\begin{pmatrix}-5 & 3\\ -10 & 6\end{pmatrix}$.

(ii) Evaluate $P^{-1}MP$ and hence $(P^{-1}MP)^n$.

(iii) Hence, or otherwise, show that $M^n=M$, for all $n\in\mathbb{N}$.

Solution

3 (a)

Conjugate

$\overline{\text{Re}+i\,\text{Im}}=\text{Re}-i\,\text{Im}$

If you multiply $a+ib$ by its conjugate $a-ib$ you get a^2+b^2.

Division: Multiply above and below by the conjugate of the number on the bottom.

$$\frac{1+2i}{2-i}$$

$$=\frac{(1+2i)}{(2-i)}\times\frac{(2+i)}{(2+i)}$$

$$=\frac{2+i+4i+2i^2}{4-i^2}$$

$$=\frac{2+i+4i-2}{4-(-1)}$$

$$=\frac{5i}{5}=i$$

3 (b) (i)

1. $\sqrt{-3+4i}=a+bi$

2. $-3+4i=(a^2-b^2)+2abi$

3. $-3=a^2-b^2$

$4=2ab \Rightarrow 2=ab$

4. $a=1,\ b=2$

5. $a+bi=\pm(1+2i)$

Steps

1. Put $\sqrt{a+ib}=c+id$.

2. Square: $a+ib=(c^2-d^2)+i2cd$.

3. Equate the real and imaginary parts.

4. Solve simultaneously by guessing.

5. There are two answers $(\pm)$.

3 (b) (ii)

$$x=\frac{-1\pm\sqrt{(-1)^2-4(1)(1-i)}}{2(1)}$$
$$=\frac{-1\pm\sqrt{1-4+4i}}{2}$$
$$=\frac{-1\pm\sqrt{-3+4i}}{2}$$
$$=\frac{-1\pm(1+2i)}{2}$$
$$=\frac{-1+(1+2i)}{2},\ \frac{-1-(1+2i)}{2}$$
$$=\frac{-1+1+2i}{2},\ \frac{-1-1-2i}{2}$$
$$=\frac{2i}{2},\ \frac{-2-2i}{2}$$
$$=i,\ -1-i$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=1$

$b=1$

$c=(1-i)$

3 (c) (i)

$$(A^{-1}BA)^n=\underbrace{(A^{-1}BA)(A^{-1}BA)(A^{-1}BA)\ldots\ldots\ldots\ldots(A^{-1}BA)(A^{-1}BA)}_{n\text{ times}}$$

$$\therefore (A^{-1}BA)^n=(A^{-1}B)(AA^{-1})B(AA^{-1})B(A\ldots\ldots\ldots\ldots A^{-1})B(AA^{-1})BA$$
$$=A^{-1}\underbrace{BBBB\ldots\ldots\ldots\ldots}_{n\text{ times}}BA$$

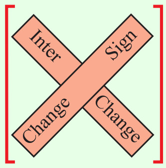
$$\therefore (A^{-1}BA)^n=A^{-1}B^nA$$

3 (c) (ii)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Remember it like this:

$$A^{-1} = \frac{1}{(ad - bc)}$$



$$P = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{6-5} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$= \frac{1}{1} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$P^{-1}MP = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1}MP = (P^{-1}MP)^n$$

$$\therefore (P^{-1}MP)^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3 (c) (iii)

$$(P^{-1}MP)^n = P^{-1}M^nP$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = P^{-1}M^nP$$

$$P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = PP^{-1}M^nPP^{-1}$$

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = M^n$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = M^n$$

$$\begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} = M^n$$

$$\therefore M^n = M$$