# Complex Numbers & Matrices (Q 3, Paper 1)

## 2010

3 (a) Find $x$ and $y$ such that

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix}.$$

(b) Let $z_1 = s + 8i$ and $z_2 = t + 8i$, where $s \in \mathbf{R}$, $t \in \mathbf{R}$, and $i^2 = -1$.

(i) Given that $|z_1| = 10$, find the possible values of $s$.

(ii) Given that $\arg(z_2) = \dfrac{3\pi}{4}$, find the value of $t$.

(c) (i) Use De Moivre's theorem to find, in polar form, the five roots of the equation

$$z^5 = 1.$$

(ii) Choose one of the roots $w$, where $w \neq 1$. Prove $w^2 + w^3$ that is real.

**Solution**

**3 (a)**

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix}$$

$3x + 4y = 20$

$5x + 6y = 32$

$$\begin{aligned} 3x + 4y &= 20 \ldots(\mathbf{1}) \\ 5x + 6y &= 32 \ldots.(\mathbf{2}) \end{aligned} \quad \rightarrow \quad \begin{aligned} 15x + 20y &= 100 \ldots(\mathbf{1})(\times 5) \\ -15x - 18y &= -96 \ldots.(\mathbf{2})(\times -3) \\ \hline 2y &= 4 \Rightarrow y = 2 \end{aligned}$$

Substitute this value of $y$ into Eqn. (**1**):

$3x + 4(2) = 20$

$3x + 8 = 20$

$3x = 13$

$x = 4$

**3 (b) (i)**

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{\text{Re}^2 + \text{Im}^2}$$

$|z_1| = 10 \Rightarrow |s + 8i| = 10$

$\sqrt{s^2 + 8^2} = 10$

$s^2 + 64 = 100$

$s^2 = 36$

$s^2 = \sqrt{36} = \pm 6$

**3 (b) (ii)**

$$\tan\theta = \frac{\text{Im}}{\text{Re}}$$

$\arg(z_2) = \dfrac{3\pi}{4} \Rightarrow \arg(t + 8i) = \dfrac{3\pi}{4}$

$\therefore \tan\left(\dfrac{3\pi}{4}\right) = \dfrac{8}{t}$

$-1 = \dfrac{8}{t}$

$t = -8$

**3 (c) (i)**

**STEPS**
**1.** Write the complex number in general polar form.
**2.** Apply De Moivre's Theorem.
**3.** List all roots (start at $n$ = 0) changing nice angles to Cartesian form.

**STEP 1.**

$z^5 = 1 = 1 + 0i$

$\therefore z = (1+0i)^{\frac{1}{5}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{5}}$

**STEP 2.**

$z = (\cos \tfrac{2}{5}n\pi + i\sin \tfrac{2}{5}n\pi)$ $\quad$ $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$

**STEP 3.**

$n = 0 : z_1 = (\cos 0 + i\sin 0) = 1$

$n = 1 : z_2 = (\cos \tfrac{2}{5}\pi + i\sin \tfrac{2}{5}\pi)$

$n = 2 : z_3 = (\cos \tfrac{4}{5}\pi + i\sin \tfrac{4}{5}\pi)$

$n = 3 : z_4 = (\cos \tfrac{6}{5}\pi + i\sin \tfrac{6}{5}\pi)$

$n = 4 : z_5 = (\cos \tfrac{8}{5}\pi + i\sin \tfrac{8}{5}\pi)$

**3 (c) (ii)**

$\omega = (\cos \tfrac{2}{5}\pi + i\sin \tfrac{2}{5}\pi)$

$\omega^2 + \omega^3 = (\cos \tfrac{2}{5}\pi + i\sin \tfrac{2}{5}\pi)^2 + (\cos \tfrac{2}{5}\pi + i\sin \tfrac{2}{5}\pi)^3$

$= (\cos \tfrac{4}{5}\pi + i\sin \tfrac{4}{5}\pi) + (\cos \tfrac{6}{5}\pi + i\sin \tfrac{6}{5}\pi)$ $\quad$ $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$

$= (\cos \tfrac{10}{5}\pi + i\sin \tfrac{10}{5}\pi)$ $\quad$ $(\cos A \oplus i\sin A)(\cos B \oplus i\sin B) = \cos(A+B) + i\sin(A+B)$

$= (\cos 2\pi + i\sin 2\pi)$

$= 1$