COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

2009

3 (a) $z_1 = a + bi$ and $z_2 = c + di$, where $i^2 = -1$.

Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, where $\overline{z}$ is the complex conjugate of z.

(b) Let $A = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

(i) Express A^3 in the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbf{Z}$.

(ii) Hence, or otherwise, find A^{17}.

(c) (i) Use De Moivre's theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(ii) Hence, find $\int \sin^3\theta \, d\theta$.

SOLUTION

3 (a)

LHS

$$\overline{z_1 + z_2}$$
$$= \overline{a + bi + c + di}$$
$$= \overline{(a+c) + (b+d)i}$$
$$= (a+c) - (b+d)i$$

RHS

$$\overline{z_1} + \overline{z_2}$$
$$= \overline{a+bi} + \overline{c+di}$$
$$= a - bi + c - di$$
$$= (a+c) - (b+d)i$$

$\overline{\text{Re} + i\,\text{Im}} = \text{Re} - i\,\text{Im}$

3 (b) (i)

$$A = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$A^2 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$A^3 = A \times A^2$$

$$A^3 = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

3 (b) (ii)

$$A^{17} = A^2 \times A^{15}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{15}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} (-1)^{15} & 0 \\ 0 & (-1)^{15} \end{pmatrix}$$

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

3 (c) (i)

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$$

$$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

$$\cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i = \cos 3\theta + i\sin 3\theta$$

Line up the imaginary parts:

$$3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$3(1-\sin^2\theta)\sin\theta - \sin^3\theta = \sin 3\theta$$

$$3\sin\theta - 4\sin^3\theta = \sin 3\theta$$

3 (c) (ii)

$$3\sin\theta - 4\sin^3\theta = \sin 3\theta$$

$$3\sin\theta - \sin 3\theta = 4\sin^3\theta$$

$$\tfrac{1}{4}(3\sin\theta - \sin 3\theta) = \sin^3\theta$$

$$\int \sin^3\theta\, d\theta = \tfrac{1}{4}\int (3\sin\theta - \sin 3\theta)\, d\theta$$

$$\int \sin(ax+b)dx = -\tfrac{1}{a}\cos(ax+b) + c$$

$$= -\tfrac{3}{4}\cos\theta - \tfrac{1}{4}\times\tfrac{1}{3}(-\cos 3\theta) + c$$

$$= -\tfrac{3}{4}\cos\theta + \tfrac{1}{12}\cos 3\theta + c$$