COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

2008

3 (a) Let A be the matrix $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

Find the matrix B, such that $AB = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$.

(b) (i) Let $z = \dfrac{5}{2+i} - 1$, where $i^2 = -1$.

Express z in the form $a + bi$ and plot it on an Argand diagram.

(ii) Use De Moivre's theorem to evaluate z^6.

(c) Prove, by induction, that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{for } n \in \mathbf{N}.$$

SOLUTION

3 (a)

To solve for an unknown matrix A isolate it by multiplying both sides by inverse matrices in the right order.

$$AB = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} \Rightarrow A^{-1}AB = A^{-1}\begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} \Rightarrow B = A^{-1}\begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$$

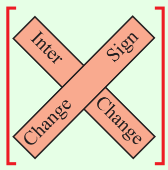
Find the inverse matrix of A.

$$A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{(3)(2)-(5)(1)}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ **8**

$$\Rightarrow A^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

Remember it like this:

$$A^{-1} = \frac{1}{(ad-bc)}$$ 

$$\therefore B = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ 5 & 0 \end{pmatrix}$$

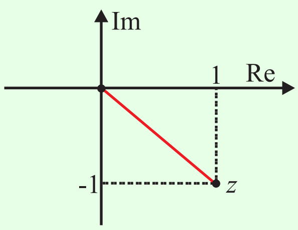
3 (b) (i)

DIVISION: Multiply above and below by the conjugate of the number on the bottom. If you multiply $a+ib$ by its conjugate $a-ib$ you get a^2+b^2.

$$z = \frac{5}{2+i} - 1 = \frac{5}{(2+i)} \times \frac{(2-i)}{(2-i)} - 1$$

$$\Rightarrow z = \frac{\cancel{5}(2-i)}{\cancel{5}} - 1 = 2 - i - 1$$

$$\therefore z = 1 - i$$



3 (b) (ii)

CHANGING FROM CARTESIAN TO POLAR

STEPS

1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.

2. Draw a free-hand picture to see what quadrant θ is in.

3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.

4. Write $z = r(\cos\theta + i\sin\theta)$.

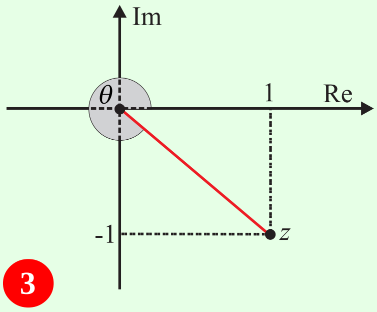
1. $z = 1 - i \Rightarrow |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ $\quad |z| = r = \sqrt{x^2 + y^2} = \sqrt{\text{Re}^2 + \text{Im}^2}$ 1

2. Picture on right.

3. $\tan\theta = \left|\frac{-1}{1}\right| = 1 \Rightarrow \theta = \tan^{-1} 1 = \frac{\pi}{4}$ $\quad \tan\theta = \frac{\text{Im}}{\text{Re}}$ 2

The angle is in the fourth quadrant.

$$\therefore \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$



4. $z = \sqrt{2}\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$ $\quad z = r(\cos\theta + i\sin\theta)$ 3

De Moivre's Theorem can be used to evaluate high whole number powers of complex numbers.

STEPS

1. Write complex number in polar form.

2. Apply De Moivre's Theorem.

3. Change to Cartesian.

1. $z = \sqrt{2}\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$

2. $z^6 = (\sqrt{2})^6\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]^6$ $\quad (\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ 4

$$\Rightarrow z^6 = 8\left[\cos\left(\frac{21\pi}{2}\right) + i\sin\left(\frac{21\pi}{2}\right)\right]$$

3. $\therefore z^6 = 8\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = 8[0 + i] = 8i$

3 (c)

STATEMENT OF DE MOIVRE'S THEOREM

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbf{N}_0$.

PROOF

1. For $n = 1$: Prove $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

i.e. $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$. This is obviously true.

2. For $n = k$: Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

3. For $n = k + 1$: Prove $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

PROOF: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ using **STEP 2**

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.