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3 (a) Given that $A=\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}3 & 1\\ -5 & -2\end{pmatrix}$, find $B^{-1}A$.

3 (b) (i) Simplify $\left(\dfrac{-2+3i}{3+2i}\right)$ and hence, find the value of $\left(\dfrac{-2+3i}{3+2i}\right)^9$ where $i^2=-1$.

(ii) Find the two complex numbers $a + ib$ such that

$$(a+ib)^2=15-8i.$$

3 (c) Use De Moivre's theorem

(i) to prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) to express $(-\sqrt{3}-i)^{10}$ in the form $2^n(1-i\sqrt{k})$ where $n, k \in \mathbf{N}$.

SOLUTION

3 (a)

$$B=\begin{pmatrix}3 & 1\\ -5 & -2\end{pmatrix}$$

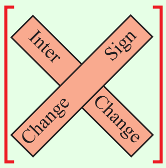
$$\Rightarrow B^{-1}=\frac{1}{3(-2)-(1)(-5)}\begin{pmatrix}-2 & -1\\ 5 & 3\end{pmatrix}$$

$$\Rightarrow B^{-1}=-1\begin{pmatrix}-2 & -1\\ 5 & 3\end{pmatrix}=\begin{pmatrix}2 & 1\\ -5 & -3\end{pmatrix}$$

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ 8

Remember it like this:

$$A^{-1}=\frac{1}{(ad-bc)}$$



$$B^{-1}A=\begin{pmatrix}2 & 1\\ -5 & -3\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}=\begin{pmatrix}4 & -1\\ -11 & 1\end{pmatrix}$$

3 (b) (i)

$\left(\dfrac{-2+3i}{3+2i}\right)$ [Multiply above and below by the conjugate of the number on the bottom.]

$$=\left(\frac{-2+3i}{3+2i}\right)\left(\frac{3-2i}{3-2i}\right)$$

If you multiply $a+ib$ by its conjugate $a-ib$ you get a^2+b^2.

$$=\frac{-6+4i+9i-6i^2}{13}$$

$$=\frac{13i}{13}=i$$

$$\left(\frac{-2+3i}{3+2i}\right)^9=i^9=i$$

Powers of i

$i=\sqrt{-1}=i$

$i^2=-1$

$i^3=-i$

$i^4=1$

$i^{power}=i^{remainder}$ when *power* is divided by 4

Powers of i repeat in groups of four. You always get one of 4 answers: $i, -1, -i, 1$

3 (b) (ii)

1. $\sqrt{15-8i} = a+ib$
2. $15-8i = (a^2-b^2)+2abi$
3. $15 = a^2-b^2$ and $-8 = 2ab \Rightarrow -4 = ab$
4. $a=-4,\ b=1$ or $a=4,\ b=-1$
5. $a+ib = \pm(4-i)$

Steps

1. Put $\sqrt{a+ib} = c+id$.

2. Square: $a+ib = (c^2-d^2)+i2cd$.

3. Equate the real and imaginary parts.

4. Solve simultaneously by guessing.

5. There are two answers $(\pm)$.

3 (c) (i)

Steps

1. Write down De Moivre's Theorem for number in multiple angle.
2. Expand out left-hand side (*LHS*).
3. Equate the real parts and the imaginary parts.
4. Tidy up left-hand side using $\cos^2\theta+\sin^2\theta$.

$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **4**

$$(\cos\theta+i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i^2\sin^2\theta) + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i = \cos 3\theta + i\sin 3\theta$$

Equate the real parts:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ [Use $\sin^2\theta = 1-\cos^2\theta$]

$$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

3 (c) (ii)

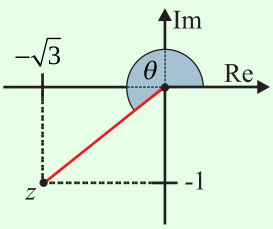
Changing from Cartesian to Polar

Steps

1. Find $r = |z| = \sqrt{\text{Re}^2+\text{Im}^2}$ first.

2. Draw a free-hand picture to see what quadrant θ is in.

3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.



1. $r = \sqrt{(-\sqrt{3})^2+(-1)^2} = \sqrt{4} = 2$
2. Draw a picture.
3. $|\tan\theta| = \left|\dfrac{-1}{-\sqrt{3}}\right| = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$

   Angle is in the third quadrant $\Rightarrow \theta = 210^\circ = 210^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{7\pi}{6}$

4. $z = 2\left\{\cos\left(\dfrac{7\pi}{6}\right) + i\sin\left(\dfrac{7\pi}{6}\right)\right\}$

STEPS
1. Write complex number in polar form.
2. Apply De Moivre's Theorem.
3. Change to Cartesian.

1. $z = 2\left\{\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right\}$

$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **4**

2. $z^{10} = 2^{10}\left\{\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right\}^{10}$

$= 2^{10}\left\{\cos\left(\frac{35\pi}{3}\right) + i\sin\left(\frac{35\pi}{3}\right)\right\}$

[**NOTE**: You can use your calculator to turn this number into its Cartesian form. Make sure your calculator in Radian mode.]

3. $z^{10} = 2^{10}\left\{\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right\}$

$= 2^{10}\{\cos 300^\circ + i\sin 300^\circ\}$

$= 2^{10}\{\cos 60^\circ - i\sin 60^\circ\}$

$= 2^{10}\{\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\}$

$= 2^{9}(1-\sqrt{3}i)$