# Complex Numbers & Matrices (Q 3, Paper 1)

## 1999

3 (a) If $A=\begin{pmatrix}2 & 1\\ 5 & 4\end{pmatrix}$, find $A^{-1}$.

3 (b) (i) Find a quadratic equation whose roots are 3 + $i$ and $3-i$, where $i^2=-1$.

(ii) Let $P(z)=z^3-kz^2+22z-20,\ k\in\mathbf{R}$.
3 + $i$ is a root of the equation $P(z)$ = 0.
Find the value of $k$.
Find the other two roots of the equation $P(z)$ = 0.

3 (c) (i) Solve for $w$

$$\sqrt{5}\,|w|+iw=3+i.$$

Write your answers in the form $u+iv,\ u,\ v\in\mathbf{R}$.

(ii) Use De Moivre's theorem to find three roots of the equation $z^6-1=0$.

**Solution**

**3 (a)**

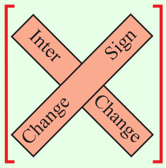
$$A=\begin{pmatrix}2 & 1\\ 5 & 4\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(2)(4)-(1)(5)}\begin{pmatrix}4 & -1\\ -5 & 2\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}4 & -1\\ -5 & 2\end{pmatrix}=\begin{pmatrix}\frac{4}{3} & -\frac{1}{3}\\ -\frac{5}{3} & \frac{2}{3}\end{pmatrix}$$

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ ....... 8

Remember it like this:

$$A^{-1}=\frac{1}{(ad-bc)}$$




**3 (b) (i)**

Quadratic: $x^2-Sz+P=0$

Roots: $3+i,\ 3-i$

**Sum S**: $3+i+3-i=6$

**Product P**: $(3+i)(3-i)=10$

Quadratic: $x^2-6z+10=0$

**Sum S**: $\alpha+\beta=-\frac{b}{a}=\frac{-2^{\text{nd.}}}{1^{\text{st.}}}$ ....... 5

**Product P**: $\alpha\beta=\frac{c}{a}=\frac{3^{\text{rd.}}}{1^{\text{st.}}}$ ....... 6

Forming a quadratic equation given its roots:

$x^2-\mathbf{S}x+\mathbf{P}=0$ ....... 7

**3 (b) (ii)**

Conjugate Root Trick: If $z$ is a root of a polynomial equation with **all real coefficients**, so is $\bar{z}$ and vice versa.

The coefficients of the cubic equation are real. Therefore, you can apply the conjugate root trick. If 3 + $i$ is a root, then $3-i$ is also a root of $P(z)$ = 0.

From part **(i)**, the quadratic equation with these roots is $z^2-6z+10=0$.

Therefore, $z^2-6z+10=0$ is a factor of $P(z)$.

There are a number of ways to find $k$. The method shown is the most efficient way.

**Lining up:**

A cubic expression is the product of a quadratic and a linear factor.
The **first** terms of the linear and quadratics multiply to give the **first** term of the cubic. The **last** terms of the linear and quadratics multiply to give the **last** term of the cubic.

$$z^3 - kz^2 + 22z - 20 = (z^2 - 6z + 10)(z - 2)$$

$$\Rightarrow z^3 - kz^2 + 22z - 20 = z^3 - 8z^2 + 22z - 10$$

$$\therefore k = 8$$

Roots: $3+i,\ 3-i,\ 2$

**3 (c) (i)**

Let $w = u + iv$

$$\sqrt{5}|w| + iw = 3 + i$$

$$\Rightarrow \sqrt{5}|u+iv| + i(u+iv) = 3+i$$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{\text{Re}^2 + \text{Im}^2}$ ....... 1

$$\Rightarrow \sqrt{5}\sqrt{u^2+v^2} + iu + i^2v = 3+i$$

$$\Rightarrow \sqrt{5}\sqrt{u^2+v^2} + iu - v = 3+i$$

$$\therefore \sqrt{5}\sqrt{u^2+v^2} - v = 3 \text{ and } iu = i$$

For all equations you can equate the real parts and the imaginary parts.

$$\therefore u = 1$$

$$\Rightarrow \sqrt{5}\sqrt{1+v^2} = v + 3$$ [Square both sides.]

$$\Rightarrow 5(1+v^2) = v^2 + 6v + 9$$

$$\Rightarrow 5 + 5v^2 = v^2 + 6v + 9$$

$$\Rightarrow 4v^2 - 6v - 4 = 0$$

$$\Rightarrow 2v^2 - 3v - 2 = 0$$

$$\Rightarrow (2v+1)(v-2) = 0$$

$$\therefore v = -\tfrac{1}{2},\ 2$$

**Ans:** $w = 1 - \tfrac{1}{2}i,\ 1 + 2i$

**3 (c) (ii)**

**Roots of Complex Numbers**: De Moivre's Theorem is used to evaluate fractional powers of complex numbers.

**Steps**
1. Write the complex number in general polar form.
2. Apply De Moivre's Theorem.
3. List all roots (start at $n = 0$) changing nice angles to Cartesian form.

$$z^6 = 1 \Rightarrow z = (1 + 0i)^{\frac{1}{6}}$$

$$\Rightarrow z = (\cos(0 + 2n\pi) + i\sin(0 + 2n\pi))^{\frac{1}{6}}$$

$$\Rightarrow z = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{6}}$$

$$\therefore z = \cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)$$

$$n = 0 \Rightarrow z_1 = \cos 0^{\circ} + i\sin 0^{\circ} = 1$$

$$n = 1 \Rightarrow z_2 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

$$n = 2 \Rightarrow z_3 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

$$n = 3 \Rightarrow z_4 = \cos(\pi) + i\sin(\pi) = -1$$

$$n = 4 \Rightarrow z_5 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

$$n = 5 \Rightarrow z_6 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$