Complex Numbers & Matrices (Q 3, Paper 1)

1998

3 (a) Express $\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.

(b) If $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$, find AB.

Show that $B^{-1}AB$ is of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where $p, q \in \mathbf{N}_0$.

(c) Let $z = \cos\theta + i\sin\theta$.

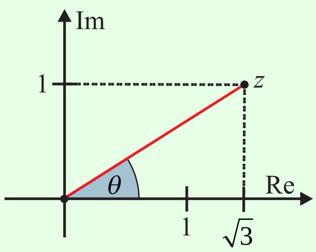
Express $\dfrac{2}{1+z}$ in the form $1 - i\tan(k\theta), k \in Q$ and $z \neq -1$.

Solution

3 (a)

Steps

1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z = r(\cos\theta + i\sin\theta)$.



1. $r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

2. Draw picture.

3. $|\tan\theta| = \left|\dfrac{1}{\sqrt{3}}\right| = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ = 30^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi}{6}$

4. $\therefore z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

3 (b)

$$AB = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 24 & 1 \\ 6 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ **8**

$$B = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow B^{-1} = \frac{1}{4(-1)-(1)(1)}\begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}$$

$$\Rightarrow B^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}$$

$$\therefore B^{-1}AB = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 24 & 1 \\ 6 & -1 \end{pmatrix}$$

$$\Rightarrow B^{-1}AB = -\frac{1}{5}\begin{pmatrix} -30 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

3 (c)

$$\frac{2}{1+z}$$

$$= \frac{2}{(1+\cos\theta)+i\sin\theta}$$

$$= \frac{2}{(1+\cos\theta)+i\sin\theta} \times \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)-i\sin\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{1+2\cos\theta+1}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{2+2\cos\theta} = \frac{\cancel{2}[(1+\cos\theta)-i\sin\theta]}{\cancel{2}(1+\cos\theta)}$$

$$= \frac{(1+\cos\theta)}{(1+\cos\theta)} - \frac{\sin\theta}{(1+\cos\theta)}i$$

$$= 1 - \frac{\sin\theta}{(1+\cos\theta)}i$$

This needs to be written in the form $1 - i\tan(k\theta)$.

$$\therefore \cancel{1} - \left(\frac{\sin\theta}{1+\cos\theta}\right) = \cancel{1} - i\tan(k\theta).$$

$$\Rightarrow \frac{\sin\theta}{1+\cos\theta} = \frac{\sin k\theta}{\cos k\theta}$$

$\tan A = \dfrac{\sin A}{\cos A}$

$$\Rightarrow \sin\theta\cos k\theta = (1+\cos\theta)\sin k\theta$$

$$\Rightarrow \sin\theta\cos k\theta = \sin k\theta + \cos\theta\sin k\theta$$

$$\Rightarrow \sin\theta\cos k\theta - \cos\theta\sin k\theta = \sin k\theta$$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ **10**

$$\Rightarrow \sin(\theta - k\theta) = \sin k\theta$$

$$\therefore \theta - k\theta = k\theta \Rightarrow \theta = 2k\theta$$

$$\Rightarrow 1 = 2k$$

$$\therefore k = \tfrac{1}{2}$$

$$\therefore \frac{2}{1+z} = 1 - i\tan(\tfrac{1}{2}\theta)$$