1997

3 (a) If $A=\begin{pmatrix}3 & 2\\ 5 & 4\end{pmatrix}$ and $B=\begin{pmatrix}1 & -2\\ 3 & 4\end{pmatrix}$, find the matrix C such that $C=A(A-B)$.

(b) Let $P(z)=z^3-(10+i)z^2+(29+10i)z-29i$, where $i^2=-1$.

(i) Determine the real numbers a and b if
$P(z)=(z-i)(z^2+az+b)$.

(ii) Plot on an argand diagram the solution set of $P(z)=0$.

(c) (i) Let $\omega_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $\omega_2=(\omega_1)^2$.
Verify that
$x^2+xy+y^2=(x-\omega_1 y)(x-\omega_2 y)$, where $x,y\in\mathbf{R}$.

(ii) Express $2(1-i\sqrt{3})$ in the form $r(\cos\theta+i\sin\theta)$.
Using De Moivre's theorem find the values for
$[2(1-i\sqrt{3})]^{\frac{3}{2}}$
and write your answers in the form $p+qi,\ p,q\in\mathbf{R}$.

SOLUTION

3 (a)

$C=A(A-B)$

$\Rightarrow C=\begin{pmatrix}3 & 2\\ 5 & 4\end{pmatrix}\left(\begin{pmatrix}3 & 2\\ 5 & 4\end{pmatrix}-\begin{pmatrix}1 & -2\\ 3 & 4\end{pmatrix}\right)$

$\Rightarrow C=\begin{pmatrix}3 & 2\\ 5 & 4\end{pmatrix}\begin{pmatrix}2 & 4\\ 2 & 0\end{pmatrix}$

$\therefore C=\begin{pmatrix}10 & 12\\ 18 & 20\end{pmatrix}$

3 (b) (i)

$z^3-(10+i)z^2+(29+10i)z-29i=(z-i)(z^2+az+b)$

$\Rightarrow z^3-(10+i)z^2+(29+10i)z-29i=z^3+az^2+bz-iz^2-aiz-ib$

$\Rightarrow z^3-(10+i)z^2+(29+10i)z-29i=z^3+(a-i)z^2+(b-ai)z-ib$

$\therefore -(10+i)=a-i\Rightarrow -10-i=a-i\Rightarrow a=-10$

$\therefore -29i=-ib\Rightarrow b=29$

3 (b) (ii)

$$P(z) = z^3 - (10+i)z^2 + (29+10i)z - 29i = 0$$

$$\Rightarrow P(z) = (z-i)(z^2 - 10z + 29) = 0$$

Solve the quadratic $z^2 - 10z + 29 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ 4

$a = 1$
$b = -10$
$c = 29$

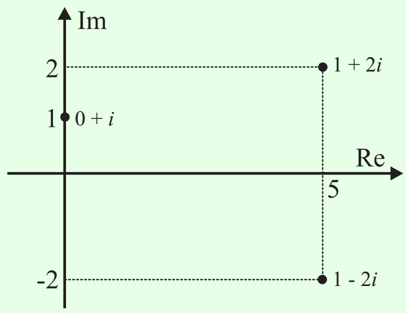
$$\therefore z = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(29)}}{2(1)}$$

$$\Rightarrow z = \frac{10 \pm \sqrt{100-116}}{2} = \frac{10 \pm \sqrt{-16}}{2}$$

$$\Rightarrow z = \frac{10 \pm 4i}{2}$$

$$\therefore z = 5 \pm 2i$$

Solutions of $P(z) = 0$: $z = i,\ 5 \pm 2i$



3 (c) (i)

$$w_1 = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i$$

$$w_2 = (w_1)^2 = (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i)^2 = (-\tfrac{1}{2})^2 + 2(-\tfrac{1}{2})(\tfrac{\sqrt{3}}{2} i) + (\tfrac{\sqrt{3}}{2} i)^2$$

$$\Rightarrow w_2 = \tfrac{1}{4} - \tfrac{\sqrt{3}}{2} i - \tfrac{3}{4}$$

$$\therefore w_2 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i$$

$$(x - w_1 y)(x - w_2 y)$$

$$= (x - (-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i)y)(x - (-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i)y)$$

$$= (x + \tfrac{1}{2}y - \tfrac{\sqrt{3}}{2} iy)(x + \tfrac{1}{2}y + \tfrac{\sqrt{3}}{2} iy)$$

$$= x^2 + \tfrac{1}{2}xy + \cancel{\tfrac{\sqrt{3}}{2} ixy} + \tfrac{1}{2}xy + \tfrac{1}{4}y^2 + \cancel{\tfrac{\sqrt{3}}{4} iy^2} - \cancel{\tfrac{\sqrt{3}}{2} ixy} - \cancel{\tfrac{\sqrt{3}}{4} iy^2} - \tfrac{3}{4} i^2 y^2$$

$$= x^2 + \tfrac{1}{2}xy + \tfrac{1}{2}xy + \tfrac{1}{4}y^2 + \tfrac{3}{4}y^2$$

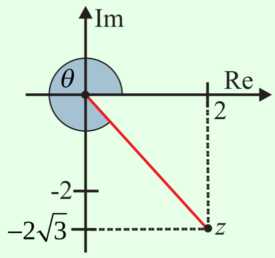
$$= x^2 + xy + y^2$$

3 (c) (ii)

Steps

1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z = r(\cos\theta + i\sin\theta)$.

1. $r=\sqrt{(2)^2+(-2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$



2. Draw a picture.

3. $|\tan\theta|=\left|\dfrac{-2\sqrt{3}}{2}\right|=\sqrt{3}\Rightarrow\theta=60^{\circ}$

Angle is in fourth quadrant $\Rightarrow\theta=300^{\circ}=300^{\circ}\times\dfrac{\pi}{180^{\circ}}=\dfrac{5\pi}{3}$

4. $\therefore z=4(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$

To find roots, write z in general polar form.

$$z=4\{\cos(\tfrac{5\pi}{3}+2n\pi)+i\sin(\tfrac{5\pi}{3}+2n\pi)\}$$

$$\Rightarrow z=4\left\{\cos\left(\frac{5\pi+6n\pi}{3}\right)+i\sin\left(\frac{5\pi+6n\pi}{3}\right)\right\}$$

Steps

1. Write the complex number in general polar form.
2. Apply De Moivre's Theorem.
3. List all roots (start at n = 0) changing nice angles to Cartesian form.

$(\cos\theta\pm i\sin\theta)^n=\cos n\theta\pm i\sin n\theta$ **4**

1. $z=4\left\{\cos\left(\dfrac{5\pi+6n\pi}{3}\right)+i\sin\left(\dfrac{5\pi+6n\pi}{3}\right)\right\}$

2. $z^{\frac{3}{2}}=4^{\frac{3}{2}}\left\{\cos\left(\dfrac{5\pi+6n\pi}{3}\right)+i\sin\left(\dfrac{5\pi+6n\pi}{3}\right)\right\}^{\frac{3}{2}}$

$$\Rightarrow z^{\frac{3}{2}}=8\left\{\cos\left(\frac{5\pi+6n\pi}{2}\right)+i\sin\left(\frac{5\pi+6n\pi}{2}\right)\right\}$$

3. $n=0: z^{\frac{3}{2}}=8\left\{\cos\left(\dfrac{5\pi+6(0)\pi}{2}\right)+i\sin\left(\dfrac{5\pi+6(0)\pi}{2}\right)\right\}=8\left\{\cos\left(\dfrac{5\pi}{2}\right)+i\sin\left(\dfrac{5\pi}{2}\right)\right\}$

$$=8\{\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}\}$$

$$=8(0+i)=0+8i$$

$n=1: z^{\frac{3}{2}}=8\left\{\cos\left(\dfrac{5\pi+6(1)\pi}{2}\right)+i\sin\left(\dfrac{5\pi+6(1)\pi}{2}\right)\right\}=8\left\{\cos\left(\dfrac{11\pi}{2}\right)+i\sin\left(\dfrac{11\pi}{2}\right)\right\}$

$$=8\{\cos\tfrac{3\pi}{2}+i\sin\tfrac{3\pi}{2}\}$$

$$=8(0-i)=0-8i$$