1996

3 (a) If $A=\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}$ and $B=\begin{pmatrix}3 & 2\\ -2 & 1\end{pmatrix}$, find a matrix M such that $M=BA^{-1}$.

(b) $P(z)=(z-2)(z^2-10z+28)$.

(i) Plot on an Argand diagram the solution set of $P(z)=0$.

(ii) Verify that the three points form an equilateral triangle.

(c) (i) $z_1=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$ and $z_2=3(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ where $i^2=-1$.

Calculate z_1z_2 in the form $x+iy$ where $x,\ y\in\mathbf{R}$.

(ii) $(2+3i)(a+ib)=-1+5i$. Express $a+ib$ in the form $r(\cos\theta+i\sin\theta)$ and hence, or otherwise, calculate $(a+ib)^{11}$.

SOLUTION

3 (a)

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ 8

$$A=\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(3)(3)-(2)(4)}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{9-8}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}=\frac{1}{1}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$$

$$\therefore A^{-1}=\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$$

$$M=BA^{-1}=\begin{pmatrix}3 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$$

$$\Rightarrow M=\begin{pmatrix}1 & 0\\ -10 & 7\end{pmatrix}$$

3 (b) (i)

$P(z) = (z-2)(z^2 - 10z + 28)$

Solve the quadratic equation $z^2 - 10z + 28 = 0.$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 4

$a = 1$
$b = -10$
$c = 28$

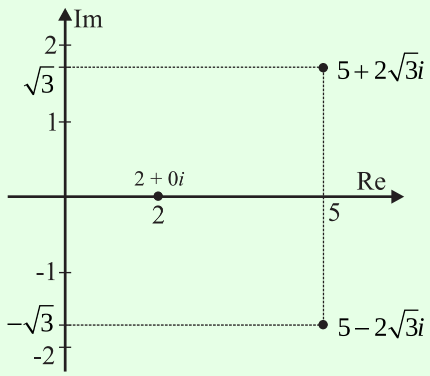
$$z = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(28)}}{2(1)}$$

$$\Rightarrow z = \frac{10 \pm \sqrt{100 - 112}}{2} = \frac{10 \pm \sqrt{-12}}{2}$$

$$\Rightarrow z = \frac{10 \pm \sqrt{12}\sqrt{-1}}{2} = \frac{10 \pm 2\sqrt{3}i}{2}$$

$$\therefore z = 5 \pm 2\sqrt{3}i$$

Solution of $P(z) = 0$: $z = 2,\ 5 \pm 2\sqrt{3}i$



3 (b) (ii)

Name the 3 points: $z_1 = 2 + 0i,\ z_2 = 5 + \sqrt{3}i,\ z_3 = 5 - \sqrt{3}i$

You need to prove that $|z_1z_2| = |z_2z_3| = |z_1z_3|$.

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$ 1

$$|z_1z_2| = \sqrt{(5-2)^2 + (\sqrt{3} - 0)^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$|z_1z_3| = \sqrt{(5-2)^2 + (-\sqrt{3} - 0)^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$|z_2z_3| = \sqrt{(5-5)^2 + (\sqrt{3} - (-\sqrt{3}))^2} = \sqrt{0 + (2\sqrt{3})^2} = 2\sqrt{3}$$

Therefore, the three points form an equilateral triangle.

3 (c) (i)

Rules for combining objects in Polar form

1. $(\cos A \oplus i \sin A)(\cos B \oplus i \sin B) = \cos(A+B) + i \sin(A+B)$

$$z_1z_2 = 6(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6})(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})$$

$$\Rightarrow z_1z_2 = 6(\cos(\tfrac{\pi}{6} + \tfrac{\pi}{3}) + i\sin(\tfrac{\pi}{6} + \tfrac{\pi}{3}))$$

$$\Rightarrow z_1z_2 = 6(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}) = 6(0 + i)$$

$$\therefore z_1z_2 = 0 + 6i$$

3 (c) (ii)

$$(2+3i)(a+ib)=-1+5i$$

$$\Rightarrow a+ib=\frac{-1+5i}{2+3i}$$

$$\Rightarrow a+ib=\frac{(-1+5i)}{(2+3i)}\times\frac{(2-3i)}{(2-3i)}$$

Division: Multiply above and below by the conjugate of the number on the bottom.

$$\Rightarrow a+ib=\frac{-2+3i+10i-15i^2}{4+9}=\frac{-2+13i+15}{13}$$

$$\therefore a+ib=\frac{13+13i}{13}=1+i$$

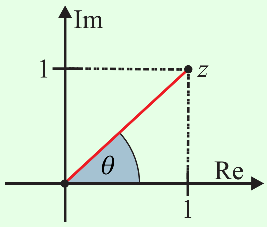
Write 1 + i in polar form.

Steps

1. Find $r=|z|=\sqrt{\text{Re}^2+\text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta|=\left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z=r(\cos\theta+i\sin\theta)$.

1. $r=|1+i|=\sqrt{1^2+1^2}=\sqrt{2}$

2. Draw a picture.



3. $|\tan\theta|=\left|\frac{1}{1}\right|=1\Rightarrow\theta=\tan^{-1}1=45^{\circ}=\frac{\pi}{4}$

4. $\therefore z=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

Now apply De Moivre's Theorem to find $(a+ib)^{11}$.

Steps

1. Write complex number in polar form.
2. Apply De Moivre's Theorem.
3. Change to Cartesian.

$$(\cos\theta\pm i\sin\theta)^n=\cos n\theta\pm i\sin n\theta \quad \text{....... } \mathbf{4}$$

1. $z=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

2. $z^{11}=2^{\frac{11}{2}}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{11}$

$\Rightarrow z^{11}=2^{\frac{11}{2}}(\cos\frac{11\pi}{4}+i\sin\frac{11\pi}{4})$

$\Rightarrow z^{11}=2^{\frac{11}{2}}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

3. $z^{11}=2^{\frac{11}{2}}(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)$

$\Rightarrow z^{11}=2^{5}(-1+i)$

$\therefore z^{11}=32(-1+i)=-32-32i$