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Circle (Q 1, Paper 2)

Lesson No. 6: Finding *g*, *f* and *c*

2005

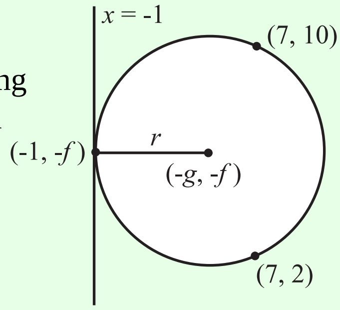
1 (c) A circle passes through the points (7, 2) and (7, 10). The line $x=-1$ is a tangent to the circle. Find the equation of the circle.

Solution

1 (c)

You need to find the values of *g*, *f* and *c* and substitute them into the equation $x^2+y^2+2gx+2fy+c=0.$

You are given two points on the circle so you can substitute them into the general equation of the circle. Sketching the circle and tangent reveals more information. You can see that the radius is the distance from the centre $(-g,-f)$ to the point of intersection $(-1,-f)$.



Equation 1: (7, 2) is on the circle.

$49+4+14g+4f+c=0 \Rightarrow 14g+4f+c=-53$ ...**(1)**

Equation 2: (7, 10) is on the circle.

$49+100+14g+20f+c=0 \Rightarrow 14g+20f+c=-149$ ...**(2)**

Equation 3: The radius is the distance from the centre $(-g,-f)$ to the point of intersection $(-1,-f)$.

$$r=\sqrt{(-g+1)^2+0^2} \Rightarrow \sqrt{g^2+f^2-c}=\sqrt{(1-g)^2}$$

$$\Rightarrow g^2+f^2-c=(1-g)^2 \Rightarrow g^2+f^2-c=1-2g+g^2 \Rightarrow 2g+f^2-c=1 \text{...}\mathbf{(3)}$$

Now look at all the equations to find *g*, *f* and *c*.

Combine equations (1) and (2):

$$\begin{aligned}14g+20f+c&=-149\text{....(2)}\\14g+4f+c&=-53\text{.......(1)}\times(-1)\end{aligned} \quad\rightarrow\quad \begin{aligned}14g+20f+c&=-149\\-14g-4f-c&=+53\\\hline 16f&=-96 \Rightarrow f=-6\end{aligned}$$

Substitute this value for *f* into equation **3** and equation **1** and solve these equations simultaneously to find *g* and *c*.

Equation **1**: $14g+4f+c=-53 \Rightarrow 14g-24+c=-53 \Rightarrow 14g+c=-29$

Equation **3**: $2g+f^2-c=1 \Rightarrow 2g+36-c=1 \Rightarrow 2g-c=-35$

$$\begin{aligned}14g+c&=-29\\2g-c&=-35\\\hline 16g&=-64 \Rightarrow g=-4\end{aligned}$$

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Substitute the values for f and g into equation **1**:

$14g + 4f + c = -53 \Rightarrow 14(-4) + 4(-6) + c = -53 \Rightarrow c = 27$

Therefore, $g = -4,\ f = -6,\ c = 27.$

Equation of the circle: $x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow x^2 + y^2 - 8x - 12y + 27 = 0$

2004

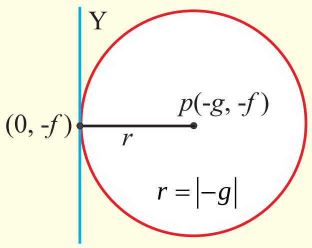
1 (c) The y-axis is a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0.$

(i) Prove that $f^2 = c.$

(ii) Find the equations of the circles that pass through the points $(-3, 6)$ and $(-6, 3)$ and have the y-axis as a tangent.

SOLUTION

1 (c) (i)



$\Rightarrow r^2 = g^2 = g^2 + f^2 - c$

$\Rightarrow \boxed{c = f^2}$

1 (c) (ii)

$(-3, 6)$ is on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 9 + 36 - 6g + 12f + c = 0 \Rightarrow 6g - 12f - c = 45 \ldots(\mathbf{1})$

$(-6, 3)$ is on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 36 + 9 - 12g + 6f + c = 0 \Rightarrow 12g - 6f - c = 45 \ldots(\mathbf{2})$

Y-axis is a tangent $\Rightarrow c = f^2 \ldots(\mathbf{3})$

Now look at the three equations. Eliminate g from equations (**1**) and (**2**).

$$\begin{aligned} 12g - 6f - c &= 45 \ldots(\mathbf{2}) \\ 6g - 12f - c &= 45 \ldots(\mathbf{1}) \times (-2) \end{aligned} \quad \rightarrow \quad \begin{aligned} 12g - 6f - c &= 45 \\ -12g + 24f + 2c &= -90 \\ \hline 18f + c &= -45 \ldots(\mathbf{4}) \end{aligned}$$

Substitute equation **3** into **4**.

$18f + c = -45 \Rightarrow 18f + f^2 = -45 \Rightarrow f^2 + 18f + 45 = 0$

$\Rightarrow (f + 15)(f + 3) = 0 \Rightarrow f = -15, -3$

Using equation **3**: $c = f^2 \Rightarrow c = 225, 9$

Using equation **1**: $6g - 12f - c = 45 \Rightarrow g = \dfrac{45 + 12f + c}{6} \Rightarrow g = 15, 3$

Therefore the two equations are:

$x^2 + y^2 + 6x - 6y + 9 = 0$ and $x^2 + y^2 + 30x - 30y + 225 = 0.$

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2001

1 (c) The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points (3, 3) and (4, 1). The line $3x - y - 6 = 0$ is a tangent to the circle at (3, 3).

(i) Find the real numbers g, f and c.

(ii) Find the co-ordinates of the point on the circle at which the tangent parallel to $3x - y - 6 = 0$ touches the circle.

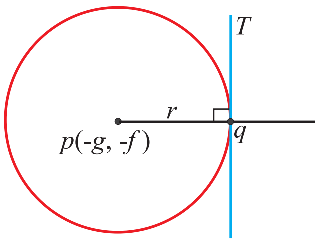
SOLUTION

1 (c) (i)

FIND THE EQUATION OF A CIRCLE GIVEN A TANGENT.

SOME POINTS TO NOTE:

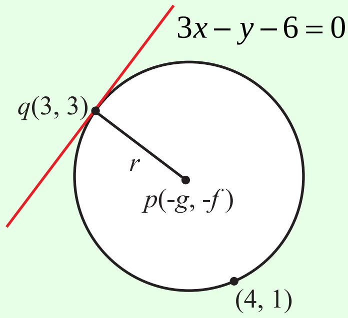
1. The tangent T is perpendicular to the line pq.
2. The perpendicular distance of the centre to the tangent T equals the radius r.
3. Distance $|pq| = r$.



Equation **1**: (3, 3) is on the circle so you can substitute it in.

$x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow 9 + 9 + 6g + 6f + c = 0$

$\Rightarrow 6g + 6f + c = -18$.....(**1**)



Equation **2**: (4, 1) is on the circle so you can substitute it in.

$x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow 16 + 1 + 8g + 2f + c = 0$

$\Rightarrow 8g + 2f + c = -17$.....(**2**)

Equation **3**: Use point No. 1 above.

Slope of $T = 3 \Rightarrow$ Slope of $pq = -\frac{1}{3}$

Slope of $pq = \dfrac{3+f}{3+g} = -\dfrac{1}{3} \Rightarrow 9 + 3f = -3 - g \Rightarrow g + 3f = -12$....(**3**)

Eliminate c from equations **1** and **2**:

$6g + 6f + c = -18$...(**1**)$\times(-1)$
$8g + 2f + c = -17$...(**2**)

$\rightarrow$

$-6g - 6f - c = +18$
$8g + 2f + c = -17$
$2g - 4f = 1$....(**4**)

Eliminate g from equations **3** and **4**.

$g + 3f = -12$...(**3**)$\times(-2)$
$2g - 4f = 1$...(**4**)

$\rightarrow$

$-2g - 6f = 24$
$2g - 4f = 1$
$-10f = 25 \Rightarrow f = -\frac{5}{2}$

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Substitute this value of *f* into equation **3** to find *g*:

$g+3f=-12 \Rightarrow g+3(-\frac{5}{2})=-12 \Rightarrow g=-12+\frac{15}{2}=-\frac{9}{2}$

Substitute these values of *g* and *f* into equation **1** to find *c*:

$6g+6f+c=-18 \Rightarrow 6(-\frac{9}{2})+6(-\frac{5}{2})+c=-18$

$\Rightarrow -27-15+c=-18 \Rightarrow c=24$

Ans: $g=-\frac{9}{2},\ f=-\frac{5}{2},\ c=24$

1 (c) (ii)

The point *q* is translated through *p* and on to *r*.

$(3,\ 3) \rightarrow (\frac{9}{2},\ \frac{5}{2}) \rightarrow (6,\ 2)$

Ans: (6, 2)