CIRCLE (Q 1, PAPER 2)

2011

1. (a) The following parametric equations define a circle:

$$x = 2 + 3\sin\theta,\ y = 3\cos\theta \quad \text{where } \theta \in \mathbb{R}.$$

What is the Cartesian equation of the circle?

(b) Find the equation of the circle that passes through the points (0, 3), (2, 1) and (6, 5).

(c) The circle c_1: $x^2 + y^2 - 8x + 2y - 23 = 0$ has centre A and radius r_1.

The circle c_2: $x^2 + y^2 + 6x + 4y + 3 = 0$ has centre B and radius r_2.

(i) Show that c_1 and c_2 intersect at two points.

(ii) Show that the tangents to c_1 at these points of intersection pass through the centre of c_2.

SOLUTION

1 (a)

$x = 2 + 3\sin\theta \Rightarrow (x-2) = 3\sin\theta$

$y = 3\cos\theta$

$\therefore (x-2)^2 + y^2 = 9\sin^2\theta + 9\cos^2\theta$

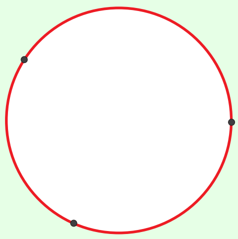
$(x-2)^2 + y^2 = 9(\sin^2\theta + \cos^2\theta)$

$(x-2)^2 + y^2 = 9$

STEPS
1. Isolate the trig functions.
2. Square both sides.
3. Add.
4. Put $\cos^2 t + \sin^2 t = 1$.

1 (b)

STEPS
1. Substitute in each point into the equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in turn and tidy up.
2. Solve them simultaneously by eliminating c from two pairs of equations.



$(0,\ 3) \in c \Rightarrow (0)^2 + (3)^2 + 2g(0) + 2f(3) + c = 0$

$0 + 9 + 0 + 6f + c = 0$

$6f + c = -9$....**(1)**

$(2,\ 1) \in c \Rightarrow (2)^2 + (1)^2 + 2g(2) + 2f(1) + c = 0$

$4 + 1 + 4g + 2f + c = 0$

$4g + 2f + c = -5$....**(2)**

$(6,\ 5) \in c \Rightarrow (6)^2 + (5)^2 + 2g(6) + 2f(5) + c = 0$

$36 + 25 + 12g + 10f + c = 0$

$12g + 10f + c = -61$....**(3)**

Subtract equations (**3**) and (**2**) to eliminate c:

$$\begin{aligned} 12g+10f+c &= -61....(\mathbf{3}) \\ 4g+2f+c &= -5......(\mathbf{2}) \\ \hline 8g+8f &= -56 \Rightarrow g+f=-7.....(\mathbf{4}) \end{aligned}$$

Subtract equations (**1**) and (**2**) to eliminate c:

$$\begin{aligned} 6f+c &= -9......(\mathbf{1}) \\ 4g+2f+c &= -5......(\mathbf{2}) \\ \hline -4g+4f &= -4 \Rightarrow -g+f=-1...(\mathbf{5}) \end{aligned}$$

Add equations (**4**) and (**5**) to calculate f:

$$\begin{aligned} g+f &= -7...(\mathbf{4}) \\ -g+f &= -1...(\mathbf{5}) \\ \hline 2f &= -8 \Rightarrow f=-4 \end{aligned}$$

Substitute this value of f into Eqn (**4**): $f=-4: g+(-4)=-7 \Rightarrow g=-3$

Substitute this value of f into Eqn (**1**): $f=-4: 6(-4)+c=-9$

$$-24+c=-9$$

$$\therefore c=15$$

Replace g, f and c by their values in the general equation of the circle:

$$c: x^2+y^2+2gx+2fy+c=0$$

$$x^2+y^2+2(-3)x+2(-4)y+15=0$$

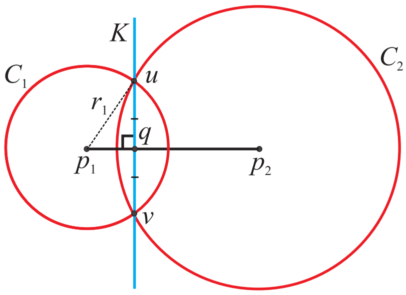
$$x^2+y^2-6x-8y+15=0$$

1 (c) (i)

Find the common chord between the circles by subtracting their equations.

Some points to note:

1. The line between the centres is perpendicular to the common chord K and bisects $[uv]$.
2. $\{u, v\} = K \cap C_1$



$$c_1: x^2+y^2-8x+2y-23=0$$

$$c_2: x^2+y^2+6x+4y+3=0$$

$$-14x-2y-26=0 \Rightarrow 7x+y+13=0 \quad \text{[Equation of the common chord]}$$

$$\therefore y=(-7x-13)$$

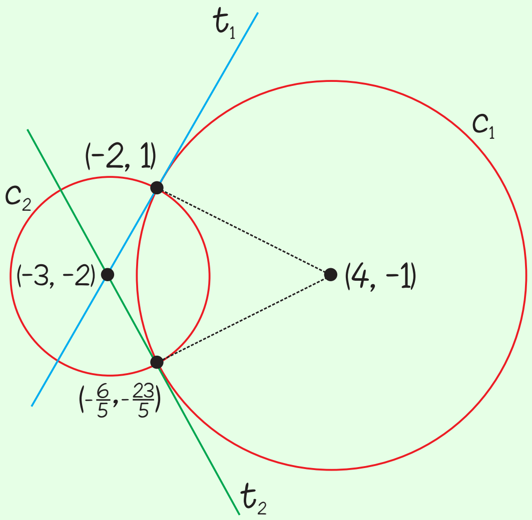
Substitute this value of y back into the equation of c_2:

$$x^2+(-7x-13)^2+6x+4(-7x-13)+3=0$$
$$x^2+49x^2+182x+169+6x-28x-52+3=0$$
$$50x^2+160x+120=0$$
$$5x^2+16x+12=0$$
$$(5x+6)(x+2)=0$$
$$\therefore x=-\tfrac{6}{5},\ -2$$

$$x=-\tfrac{6}{5}: y=-7(-\tfrac{6}{5})-13=\tfrac{42}{5}-13=-\tfrac{23}{5}$$
$$x=-2: y=-7(-2)-13=14-13=1$$

Points of intersection: $(-\tfrac{6}{5},-\tfrac{23}{5}),\ (-2,1)$

1 (c) (ii)
Call t_1 and t_2, the tangents to c_1 at these points of intersection.



$c_1: x^2+y^2-8x+2y-23=0$
Centre: $(-g,-f)=(4,-1)$
$c_2: x^2+y^2+6x+4y+3=0$
Centre: $(-g,-f)=(-3,-2)$

EQUATION OF TANGENT t_1
Find the slope between the point of contact $(-2, 1)$ and the centre of c_1 $(4,-1)$.

$$m_1=\frac{1-(-1)}{-2-4}=\frac{2}{-6}=-\frac{1}{3}$$

Slope of t_1 is perpendicular to this slope: $m_1^{\perp}=3$

Equation of t_1: $t_1: 3x-y+k=0$

$$(-2,1)\in t_1 \Rightarrow 3(-2)-(1)+k=0$$
$$-6-1+k=0$$
$$\therefore k=7$$
$$t_1: 3x-y+7=0$$

Is $(-3,-2)$ on t_1? $3(-3)-(-2)+7$
$=-9+2+7$
$=0$ [Therefore, tangent t_1 passes through the centre of c_2.]

EQUATION OF TANGENT t_2

Find the slope between the point of contact $(-\frac{6}{5}, -\frac{23}{5})$ and the centre of c_1 $(4, -1)$.

$$m_2 = \frac{-\frac{23}{5} - (-1)}{-\frac{6}{5} - 4} = \frac{-\frac{23}{5} + 1}{-\frac{6}{5} - 4} = \frac{-\frac{18}{5}}{-\frac{26}{5}} = \frac{18}{26} = \frac{9}{13}$$

Slope of t_2 is perpendicular to this slope: $m_2^{\perp} = -\frac{13}{9}$

Equation of t_2: $t_2: 13x + 9y + k = 0$

$$(-\tfrac{6}{5}, -\tfrac{23}{5}) \in t_2 \Rightarrow 13(-\tfrac{6}{5}) + 9(-\tfrac{23}{5}) + k = 0$$
$$-\tfrac{78}{5} - \tfrac{207}{5} + k = 0$$
$$-\tfrac{285}{5} + k = 0$$
$$-57 + k = 0$$
$$\therefore k = 57$$
$$t_2: 13x + 9y + 57 = 0$$

Is $(-3, -2)$ on t_2? $13(-3) + 9(-2) + 57$

$= -39 - 18 + 57$

$= 0$ [Therefore, tangent t_2 passes through the centre of c_2.]

ALTERNATIVE SOLUTION:

1 (c) (i)

c_1: $x^2 + y^2 - 8x + 2y - 23 = 0 : A(4, -1)$, $r_1 = \sqrt{4^2 + (-1)^2 - (-23)} = \sqrt{40} = 2\sqrt{10}$

c_2: $x^2 + y^2 + 6x + 4y + 3 = 0 : B(-3, -2)$, $r_2 = \sqrt{(-3)^2 + (-2)^2 - 3} = \sqrt{10}$

The circles intersect at two points if the distance between their centres is less than the sum of the radii: $|AB| < r_1 + r_2$.

$|AB| = \sqrt{(4+3)^2 + (-1+2)^2} = \sqrt{49+1} = \sqrt{50}$

$r_1 + r_2 = 2\sqrt{10} + \sqrt{10} = 3\sqrt{10}$

$\therefore |AB| < r_1 + r_2 \Rightarrow$ circles intersect at two points.

1 (c) (ii)

You need to show that the tangent t_1 to the circle c_1 at the point of intersection P passes through the centre of c_2.

AP is perpendicular to t_1 and so if the above statement is true then triangle ABP is a right-angled triangle.

$(\sqrt{10})^2 + (\sqrt{40})^2 = (\sqrt{50})^2$?

$10 + 40 = 50$ (True)

The same applies to the other point of intersection Q.