CIRCLE (Q 1, PAPER 2)

2009

1 (a) Show that, for all values of $t \in \mathbf{R}$, the point $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ lies on the circle

$x^2 + y^2 = 1$.

(b) (i) Find the equation of the tangent to the circle $x^2 + y^2 = 10$ at the point (3, 1).

(ii) Find the values of $k \in \mathbf{R}$ for which the line $x - y + k = 0$ is a tangent to the

circle $(x-3)^2 + (y+4)^2 = 50$.

(c) Two circles intersect at $p(2, 0)$ and $q(-2, 8)$. The distance from the centre of each circle to the common chord $[pq]$ is $\sqrt{20}$.
Find the equations of the two circles.

SOLUTION

1 (a)

Substitute the point into the equation of the circle:

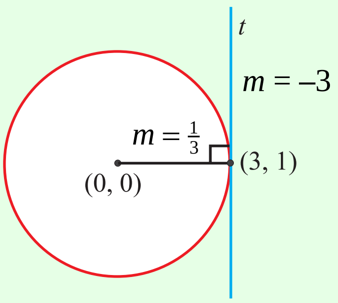
$$x^2 + y^2$$

$$= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 = \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{4t^2 + (1-t^2)^2}{(1+t^2)^2} = \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2}$$

$$= \frac{1 + 2t^2 + t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

1 (b) (i)



Circle: $x^2 + y^2 = 10$: Centre (0, 0), $r = \sqrt{10}$

Slope between centre and point of contact: $m = \frac{1}{3}$

Perpendicular slope of tangent t: $m = -3$

Equation of t: Point (3, 1), $m = -3$

$$t: 3x + y + k = 0$$

$$(3, 1) \in t \Rightarrow 3(3) + (1) + k = 0$$

$$9 + 1 + k = 0$$

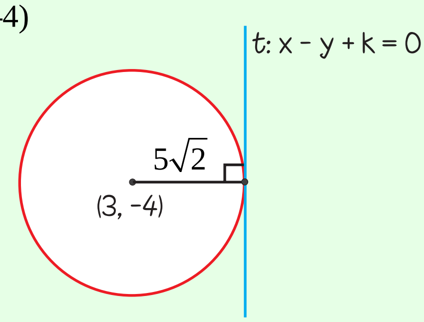
$$k = -10$$

$$t: 3x + y - 10 = 0$$

1 (b) (ii)

Circle: $(x-3)^2+(y+4)^2=50 : r=\sqrt{50}=5\sqrt{2}$, centre (3, –4)

Perpendicular distance of centre to tangent t is equal to the radius.



$$\boxed{d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$$

$$5\sqrt{2}=\frac{|(3)-(-4)+k|}{\sqrt{1^2+(-1)^2}}$$

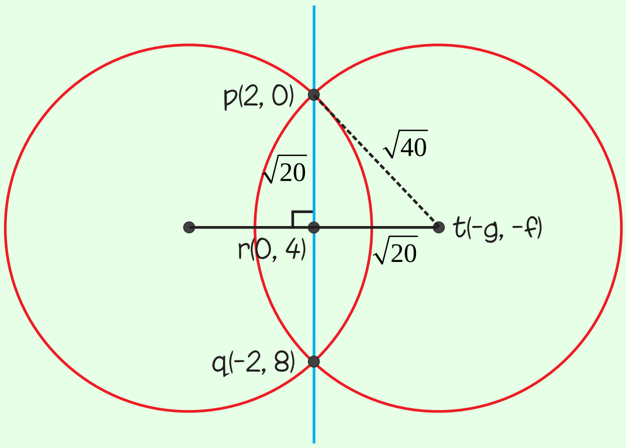
$$5\sqrt{2}=\frac{|3+4+k|}{\sqrt{2}}$$

$$10=|7+k|$$

$$\pm 10=7+k$$

$7+k=10$	$7+k=-10$
$k=3$	$k=-17$

1 (c)



Common chord is perpendicular to line joining the centres.
Call r the midpoint of pq. Midpoint of pq: $r(0, 4)$
Call t the centre of the circle: $t(-g, -f)$

Slope of $pr = \dfrac{0-4}{2-0}=\dfrac{-4}{2}=-2$

Perpendicular slope of $rt = \frac{1}{2}$

Slope of rt:

$$\frac{-f-4}{-g-0}=\frac{1}{2}\Rightarrow\frac{f+4}{g}=\frac{1}{2}$$

$2f+8=g$...**(1)**

$|rt| = \sqrt{20}$

$|pr| = \sqrt{(2-0)^2 + (0-4)^2} = \sqrt{4+16} = \sqrt{20}$

$|pt|^2 = |pr|^2 + |rt|^2$

$\quad = 20 + 20 = 40$

$\therefore |pt| = \sqrt{40}$

Distance *pt*: $\sqrt{(-g-0)^2 + (-f-4)^2} = \sqrt{20}$

$(-g-0)^2 + (-f-4)^2 = 20$

$g^2 + f^2 + 8f + 16 = 20$

$g^2 + f^2 + 8f = 4$....(**2**)

Replace g in Eqn. (**2**) by its value from Eqn. (**1**):

$(2f+8)^2 + f^2 + 8f = 4$

$4f^2 + 32f + 64 + f^2 + 8f - 4 = 0$

$5f^2 + 40f + 60 = 0$

$f^2 + 8f + 12 = 0$

$(f+6)(f+2) = 0$

$f = -6, -2$

$f = -6 : g = 2(-6) + 8 = -4 \Rightarrow (-g, -f) = (4, 6)$ is a centre.

$f = -2 : g = 2(-2) + 8 = 4 \Rightarrow (-g, -f) = (-4, 2)$ is a centre.

Circle 1: Centre (4, 6), $r = \sqrt{40}$

Equation of circle 1:

$(x-4)^2 + (y-6)^2 = 40$

$x^2 - 8x + 16 + y^2 - 12y + 36 = 40$

$x^2 + y^2 - 8x - 12y + 12 = 0$

Circle 2: Centre (–4, 2), $r = \sqrt{40}$

Equation of circle 2:

$(x+4)^2 + (y-2)^2 = 40$

$x^2 + 8x + 16 + y^2 - 4y + 4 = 40$

$x^2 + y^2 + 8x - 4y - 20 = 0$