CIRCLE (Q 1, PAPER 2)

LESSON NO. 2: LINE AND CIRCLE

2006

1 (c) S is the circle $x^2 + y^2 + 4x + 4y - 17 = 0$ and K is the line $4x + 3y = 12$.

(i) Show that the line K does not intersect S.

(ii) Find the co-ordinates of the point on S that is closest to K.

SOLUTION

1 (c) (i)

To show that the line K does not intersect the circle S can be done in two ways:
Method 1: Solve K and S simultaneously and show it has no real solutions.
Method 2: Show that the perpendicular distance from the centre of the circle to the line K is greater than the radius of the circle. [This is a better method.]

Method 1: K: $4x + 3y = 12 \Rightarrow x = \dfrac{12 - 3y}{4}$

S: $x^2 + y^2 + 4x + 4y - 17 = 0 \Rightarrow \left(\dfrac{12-3y}{4}\right)^2 + y^2 + 4\left(\dfrac{12-3y}{4}\right) + 4y - 17 = 0$

$\Rightarrow \left(\dfrac{144 - 72y + 9y^2}{16}\right) + y^2 + 12 - 3y + 4y - 17 = 0$

$\Rightarrow 144 - 72y + 9y^2 + 16y^2 + 192 - 48y + 64y - 272 = 0$

$\Rightarrow 25y^2 - 56y + 64 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **4**

REMEMBER: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

$a = 25,\ b = -56,\ c = 64$

$b^2 - 4ac = (-56)^2 - 4(25)(64) = 3136 - 6400 = -3264 < 0$

Therefore, there are no real solutions and so K and S do not intersect.

Method 2:

S: $x^2 + y^2 + 4x + 4y - 17 = 0$

Centre $(-2, -2)$, $r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 4 + 17} = 5$

$d = \dfrac{|4(-2) + 3(-2) - 12|}{\sqrt{4^2 + 3^2}} = \dfrac{|-26|}{5} = \dfrac{26}{5} > 5$

Circle C centre $(-g, -f)$, radius r.

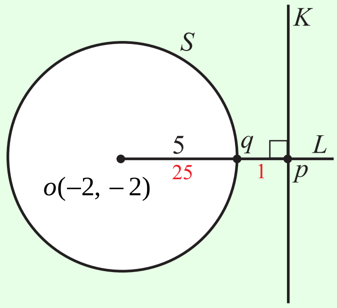
$x^2 + y^2 + 2gx + 2fy + c = 0$ **3**

$r = \sqrt{g^2 + f^2 - c}$ **4**

$d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ **8**

1 (c) (ii)

You need to find the co-ordinates of point q. There are two methods to do this. They both involve finding the equation of L, which is perpendicular to K and contains $o(-2, -2)$.



Equation of L: Point $(-2, -2)$, $m = \frac{3}{4}$

L: $3x - 4y + k = 0$

$(-2, -2) \in L \Rightarrow 3(-2) - 4(-2) + k = 0 \Rightarrow -6 + 8 + k = 0 \Rightarrow k = -2$

Equation of L: $3x - 4y - 2 = 0$

Method 1: q divides the line op in the ratio 25:1. [The distance op is $\frac{26}{25}$ whereas the distance oq is $5 = \frac{25}{25}$.] You need to find p by solving lines K and L simultaneously.

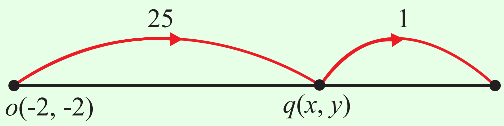
$$4x + 3y = 12 \; (\times 4)$$
$$3x - 4y = 2 \; (\times 3)$$

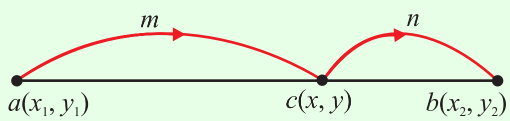
$$16x + 12y = 48$$
$$9x - 12y = 6$$
$$25x = 54 \Rightarrow x = \tfrac{54}{25}$$

$$4x + 3y = 12$$
$$\Rightarrow y = \frac{12 - 4x}{3} = \frac{12 - 4(\frac{54}{25})}{3} = \frac{28}{25}$$

Therefore, the co-ordinates of $p(\frac{54}{25}, \frac{28}{25})$.

$$x = \frac{mx_2 + nx_1}{m + n}, \; y = \frac{my_2 + ny_1}{m + n}$$ 5





$$x = \frac{25(\frac{54}{25}) + 1(-2)}{25 + 1} = \frac{54 - 2}{26} = \frac{52}{26} = 2 \text{ and } y = \frac{25(\frac{28}{25}) + 1(-2)}{25 + 1} = \frac{28 - 2}{26} = \frac{26}{26} = 1$$

Therefore, the co-ordinates of $q(2, 1)$.

Method 2: Intersect line L with circle S by solving simultaneously. There will be two solutions. q is the point closer to p.

$$L: 3x - 4y = 2 \Rightarrow x = \frac{4y + 2}{3}$$

$$S: \; x^2 + y^2 + 4x + 4y - 17 = 0 \Rightarrow \left(\frac{4y + 2}{3}\right)^2 + y^2 + 4\left(\frac{4y + 2}{3}\right) + 4y - 17 = 0$$

$$\Rightarrow \left(\frac{16y^2 + 16y + 4}{9}\right) + y^2 + \left(\frac{16y + 8}{3}\right) + 4y - 17 = 0$$

$$\Rightarrow 16y^2 + 16y + 4 + 9y^2 + 48y + 24 + 36y - 153 = 0$$

$$\Rightarrow 25y^2 + 100y - 125 = 0 \Rightarrow y^2 + 4y - 5 = 0$$

$$\Rightarrow (y + 5)(y - 1) = 0 \Rightarrow y = -5, 1 \Rightarrow x = -6, 2$$

Therefore, the points of intersection are: $(-6, -5), (2, 1)$

The answer is (2, 1) as it is closer to the line. You can check by using the perpendicular distance formula.

2004

1 (b) The point $a(5, 2)$ is on the circle K: $x^2 + y^2 + px - 2y + 5 = 0$.

(i) Find the value of p.

(ii) The line L: $x - y - 1 = 0$ intersects the circle K. Find the co-ordinates of the points of intersection.

SOLUTION

1 (b) (i)

If a point is on the circle you can substitute it into the circle equation.

$\therefore 25 + 4 + 5p - 4 + 5 = 0 \Rightarrow 5p = -30 \Rightarrow p = -6$

1 (b) (ii)

STEPS
1. Isolate x or y using equation of the line.
2. Substitute into the equation of the circle and solve simultaneously.

1. L: $x - y - 1 = 0 \Rightarrow x = y + 1$

2. K: $x^2 + y^2 - 6x - 2y + 5 = 0 \Rightarrow (y+1)^2 + y^2 - 6(y+1) - 2y + 5 = 0$

$\Rightarrow y^2 + 2y + 1 + y^2 - 6y - 6 - 2y + 5 = 0 \Rightarrow 2y^2 - 6y = 0$

$\Rightarrow y^2 - 3y = 0 \Rightarrow y(y-3) = 0 \Rightarrow y = 0, 3 \Rightarrow x = 1, 4$

Ans: Points of intersection are (1, 0) and (4, 3).

2001

1 (b) The equation of a circle is $(x+1)^2+(y-8)^2=160$. The line $x-3y+25=0$ intersects the circle at the points p and q.

(i) Find the co-ordinates of p and the co-ordinates of q.

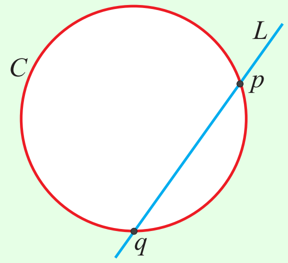
(ii) Investigate if $[pq]$ is a diameter of the circle.

SOLUTION

1 (b) (i)

The line L intersects the circle C at two points, p and q. To find p and q follow the steps.

STEPS
1. Isolate x or y using equation of the line.
2. Substitute into the equation of the circle and solve simultaneously.



L: $x-3y+25=0 \Rightarrow x=3y-25$

C: $(x+1)^2+(y-8)^2=160 \Rightarrow (3y-25+1)^2+(y-8)^2=160$

$\Rightarrow (3y-24)^2+(y-8)^2=160 \Rightarrow (3[y-8])^2+(y-8)^2=160$

$\Rightarrow 9(y-8)^2+(y-8)^2=160 \Rightarrow 10(y-8)^2=160$

$\Rightarrow (y-8)^2=16 \Rightarrow y-8=\pm 4 \Rightarrow y=4,\ 12 \Rightarrow x=-13,\ 11$

Ans: $p(-13, 4)$, $q(11, 12)$

1 (b) (ii)

$[pq]$ is a diameter if the midpoint of $[pq]$ is the centre of the circle

OR

the length of $[pq]$ equals the diameter (twice the radius) of the circle.

Midpoint of $[pq]$: $\left(\frac{-13+11}{2}, \frac{4+12}{2}\right)=(-1, 8)$

Centre of circle: $(-1, 8)$

Therefore, $[pq]$ is the diameter.

OR

$|pq|=\sqrt{(-13-11)^2+(4-12)^2}=\sqrt{576+64}=\sqrt{640}=8\sqrt{10}$

Radius of circle $r=\sqrt{160}=4\sqrt{10} \Rightarrow 2r=8\sqrt{10}$

Therefore, $[pq]$ is the diameter.