Circle (Q 1, Paper 2)

1999

1 (a) Find the Cartesian equation of the circle

$x = 6 + \cos\theta,\ y = 4 + \sin\theta,$

where $0 \le \theta \le 2\pi$.

1 (b) The equation of a circle with radius length 7 is

$x^2 + y^2 - 10kx + 6y + 60 = 0$ where $k > 0$.

(i) Find the centre of the circle in terms of k.

(ii) Find the value of k.

(iii) The line $3x + 4y + d = 0$ is a tangent to the circle, where $d \in \mathbf{Z}$.
Show that one value for d is 17.
Find the other value for d.

1 (c) Two circles intersect at the points $a(1, 2)$ and $b(7, -6)$. The line joining the centres of the circles is the perpendicular bisector of [ab].
The distance form the centre of each circle to the midpoint of [ab] is 10.
Find the midpoint of [ab] and the radius length of each circle.
Find the equation of each circle.

Solution

1 (a)

1. $x = 6 + \cos\theta \Rightarrow (x - 6) = \cos\theta$

$y = 4 + \sin\theta \Rightarrow (y - 4) = \sin\theta$

2. $(x - 6)^2 = \cos^2\theta$

$(y - 4)^2 = \sin^2\theta$

3. $(x - 6)^2 + (y - 4)^2 = \cos^2\theta + \sin^2\theta$

4. $\therefore (x - 6)^2 + (y - 4)^2 = 1$

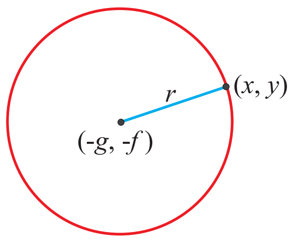
Steps
1. Isolate the trig functions.
2. Square both sides.
3. Add.
4. Put $\cos^2 t + \sin^2 t = 1$.

1 (b) (i)

Circle C with centre $(-g, -f)$, radius r.

$x^2 + y^2 + 2gx + 2fy + c = 0$ **3**

$r = \sqrt{g^2 + f^2 - c}$ **4**



$x^2 + y^2 - 10kx + 6y + 60 = 0$

$\therefore$ Centre $(5k, -3)$

1 (b) (ii)

Centre $(5k, -3)$, $r = 7$

$\Rightarrow \sqrt{25k^2 + 9 - 60} = 7$

$\Rightarrow 25k^2 - 51 = 49$

$\Rightarrow 25k^2 = 100$

$\Rightarrow k^2 = 4$

$\therefore k = 2$ as $k > 0$

1 (b) (iii)

The perpendicular distance from the centre of the circle C to the tangent T equals the radius r.

Centre $(10, -3)$, $r = 7$

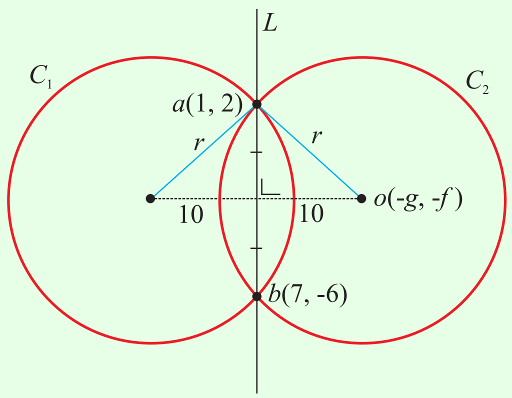
$\therefore 7 = \frac{|3(10) + 4(-3) + d|}{\sqrt{3^2 + 4^2}} \Rightarrow 7 = \frac{|30 - 12 + d|}{5}$ $\quad d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ 8

$\Rightarrow 35 = |d + 18|$

$\Rightarrow d + 18 = \pm 35$

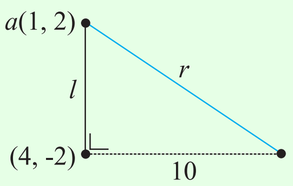
$\therefore d = 17, -53$

1 (c)



Midpoint of $[ab] = \left(\frac{1+7}{2}, \frac{2-6}{2}\right) = (4, -2)$

To find the radius, consider one of the right-angled triangles.



$l = \sqrt{(1-4)^2 + (2-(-2))^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\therefore r^2 = 5^2 + 10^2 = 125$

$\therefore r = \sqrt{125} = 5\sqrt{5}$

Finding the equations of the circles: $x^2 + y^2 + 2gx + 2fy + c = 0$ 3

$a(1, 2) \in C \Rightarrow 1 + 4 + 2g + 4f + c = 0$

$\therefore 2g + 4f + c = -5$.....**(1)**

$b(7, -6) \in C \Rightarrow 49 + 36 + 14g - 12f + c = 0$

$\therefore 14g - 12f + c = -85$...**(2)**

The perpendicular distance from the centre of the circle to the line L equals 10.

First, find the equation of L. Find the slope of ab.

$a(1,\ 2),\ b(7,\ -6)$

$$m=\frac{-6-2}{7-1}=\frac{-8}{6}=-\frac{4}{3}$$

$$m=\tan\theta=\frac{y_2-y_1}{x_2-x_1}$$ **2**

Equation of L:

$L: 4x+3y+k=0$

$a(1,\ 2)\in L \Rightarrow 4(1)+3(2)+k=0 \Rightarrow 4+6+k=0$

$\therefore k=-10$

$L: 4x+3y-10=0$

Point $(-g,\ -f)$, $L: 4x+3y-10=0,\ d=10$

$$\therefore 10=\frac{|-4g-3f-10|}{\sqrt{4^2+3^2}} \Rightarrow 10=\frac{|-4g-3f-10|}{\sqrt{25}}$$

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$ **8**

$\Rightarrow 50=|-4g-3f-10|$

$\Rightarrow 50=|4g+3f+10|$

$\Rightarrow \pm 50=4g+3f+10$

$\therefore 40=4g+3f$...Eqn (**3a**) and $-60=4g+3f$...Eqn (**3b**)

Combine Eqns. (**1**) and (**2**):

$$\begin{array}{l} 2g+4f+c=-5.....(\mathbf{1}) \\ 14g-12f+c=-85...(\mathbf{2}) \\ \hline -12g+16f=80 \Rightarrow -3g+4f=20....(\mathbf{4}) \end{array}$$

Now combine Eqn (**4**) with each of the Eqns. (**3a**) and (**3b**) to solve for g and f for each circle.

$$\begin{array}{l} 4g+3f=40....(\mathbf{3a})(\times 3) \\ -3g+4f=20....(\mathbf{4})(\times 4) \end{array} \rightarrow \begin{array}{l} 12g+9f=120 \\ -12g+16f=80 \\ \hline 25f=200 \Rightarrow f=8 \end{array}$$

Substitute this value of f into Eqn. (**4**):

$\therefore -3g+4(8)=20 \Rightarrow -3g+32=20$

$\Rightarrow -3g=-12 \Rightarrow g=4$

Substitute these values of g and f into Eqn. (**1**):

$\therefore 2(4)+4(8)+c=-5 \Rightarrow 8+32+c=-5 \Rightarrow c=-45$

Equation of C_1: $x^2+y^2+8x+16y-45=0$

$$\begin{array}{l} 4g+3f=-60....(\mathbf{3b})(\times 3) \\ -3g+4f=20.......(\mathbf{4})(\times 4) \end{array} \rightarrow \begin{array}{l} 12g+9f=-180 \\ -12g+16f=80 \\ \hline 25f=-100 \Rightarrow f=-4 \end{array}$$

Similarly as above: $g=-12,\ c=35$

Equation of C_2: $x^2+y^2-24x-8y+35=0$