CIRCLE (Q 1, PAPER 2)

1996

1 (a) The parametric equations of a circle are

$$x = 5 + \tfrac{\sqrt{3}}{2}\cos\theta,\ y = -3 + \tfrac{\sqrt{3}}{2}\sin\theta.$$

Find its Cartesian equation.

(b) Points $(1, -1)$, $(-6, -2)$ and $(3, -5)$ are on a circle C.
Find the equation of C.

(c) S_1: $x^2 + y^2 - 6x - 4y + 12 = 0$

S_2: $x^2 + y^2 + 10x + 4y + 20 = 0$ are two circles.

(i) Find the coordinates of their centres p and q and the lengths of their radii r_1, r_2 respectively.

(ii) Verify that the lines

L: $y - 1 = 0$ and M: $4x - 3y - 1 = 0$

are tangents to S_1.

(iii)If w is the point of intersection of L and M and $w \in [pq]$, show that

$$|pw| : |wq| = r_1 : r_2.$$

SOLUTION

1 (a)

STEPS
1. Isolate the trig functions.
2. Square both sides.
3. Add.
4. Put $\cos^2 t + \sin^2 t = 1$.

1. $x = 5 + \tfrac{\sqrt{3}}{2}\cos\theta \Rightarrow (x-5) = \tfrac{\sqrt{3}}{2}\cos\theta$

$y = -3 + \tfrac{\sqrt{3}}{2}\sin\theta \Rightarrow (y+3) = \tfrac{\sqrt{3}}{2}\sin\theta$

2. $(x-5)^2 = \tfrac{3}{4}\cos^2\theta$

$(y+3)^3 = \tfrac{3}{4}\sin^2\theta$

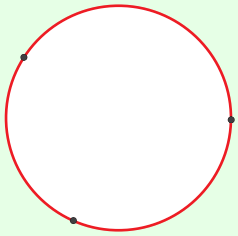
3. $(x-5)^2 + (y+3)^2 = \tfrac{3}{4}(\cos^2\theta + \sin^2\theta)$

$\Rightarrow (x-5)^2 + (y+3)^2 = \tfrac{3}{4}$

4. $\therefore 4(x-5)^2 + 4(y+3)^2 = 3$

1 (b)

Steps
1. Substitute in each point into the equation of the circle $x^2+y^2+2gx+2fy+c=0$ in turn and tidy up.
2. Solve them simultaneously by eliminating c from two pairs of equations.



$x^2+y^2+2gx+2fy+c=0$ **3**

1. $(1,-1)\in C \Rightarrow (1)^2+(-1)^2+2g(1)+2f(-1)+c=0$

$\Rightarrow 1+1+2g-2f+c=0$

$\therefore 2g-2f+c=-2.....(\mathbf{1})$

$(-6,-2)\in C \Rightarrow (-6)^2+(-2)^2+2g(-6)+2f(-2)+c=0$

$\Rightarrow 36+4-12g-4f+c=0$

$\therefore -12g-4f+c=-40.....(\mathbf{2})$

$(3,-5)\in C \Rightarrow (3)^2+(-5)^2+2g(3)+2f(-5)+c=0$

$\Rightarrow 9+25+6g-10f+c=0$

$\therefore 6g-10f+c=-34.....(\mathbf{3})$

2.

$2g-2f+c=-2.......(\mathbf{1})$
$-12g-4f+c=-40.....(\mathbf{2})$
$14g+2f\quad=38 \Rightarrow 7g+f=19...(\mathbf{4})$

$6g-10f+c=-34.....(\mathbf{3})$
$2g-2f+c=-2.......(\mathbf{1})$
$4g-8f\quad=-32 \Rightarrow g-2f=-8....(\mathbf{5})$

$7g+f=19....(\mathbf{4})(\times 2)$
$g-2f=-8...(\mathbf{5})$

$\rightarrow$

$14g+2f=38$
$g-3f=-8$
$15g\quad=30 \Rightarrow g=2$

Substitute this value of g into Eqn. (**4**).

$\therefore 7(2)+f=19 \Rightarrow 14+f=19$

$\therefore f=5$

Substitute these values of g and f into Eqn. (**1**):

$\therefore 2(2)-2(5)+c=-2 \Rightarrow 4-10+c=-2$

$\therefore c=4$

Equation of C: $x^2+y^2+4x+10y+4=0$

1 (c) (i)

$x^2+y^2+2gx+2fy+c=0$ 3

$r=\sqrt{g^2+f^2-c}$ 4

$S_1: x^2+y^2-6x-4y+12=0$

$p(3,\ 2),\ r_1=\sqrt{9+4-12}=\sqrt{1}=1$

$S_2: x^2+y^2+10x+4y+20$

$q(-5,\ -2),\ r_2=\sqrt{25+4-20}=\sqrt{9}=3$

1 (c) (ii)

Some information about Tangents:
1. A tangent *T* intersects a circle *C* at one point only, the point of contact *p*.
2. The perpendicular distance from the centre of the circle *C* to the tangent *T* equals the radius *r*.

You can use either of the above points to show the lines are tangent. I'll do one for each line.

$L: y-1=0 \Rightarrow y=1$

$\therefore S_1: x^2+(1)^2-6x-4(1)+12=0$

$\Rightarrow x^2+1-6x-4+12=0$

$\Rightarrow x^2-6x+9=0$

$\Rightarrow (x-3)(x-3)=0$

$\therefore x=3$

As there is only one solution, *L* is a tangent to S_1.

$M: 4x-3y-1=0,\ (x_1,\ y_1)=(3,\ 2)$

$d=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ 8

$$\therefore d=\frac{|4(3)-3(2)-1|}{\sqrt{4^2+(-3)^2}}=\frac{|5|}{\sqrt{16+9}}=\frac{5}{\sqrt{25}}=\frac{5}{5}=1$$

The perpendicular distance from the centre of S_1 to *M* is equal to its radius. Therefore, *M* is a tangent to S_1.

1 (c) (iii)

Solve *L* and *M* simultaneously.

$L: y=1$

$M: 4x-3y-1=0 \Rightarrow 4x-3(1)-1=0$

$\Rightarrow 4x=4 \Rightarrow x=1$

$\therefore w(1,\ 1)$ is the point of intersection.

$|pw|=\sqrt{(3-1)^2+(2-1)^2}=\sqrt{4+1}=\sqrt{5}$

$|qw|=\sqrt{(-5-1)^2+(-2-1)^2}=\sqrt{36+9}=\sqrt{45}=3\sqrt{5}$

$\therefore |pw|:|qw|=\sqrt{5}:3\sqrt{5}=1:3=r_1:r_2$