CIRCLE (Q 1, PAPER 2)

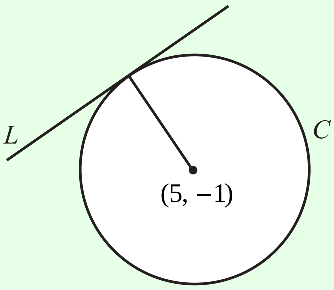
2006

1 (a) $a(-1, -3)$ and $b(3, 1)$ are the end-points of a diameter of a circle. Write down the equation of a circle.

1 (b) Circle C has centre $(5, -1)$. The line L: $3x-4y+11=0$ is a tangent to C.

(i) Show that the radius of C is 6.

(ii) The line $x + py + 1 = 0$ is also a tangent to C. Find two possible values of p.



1 (c) S is the circle $x^2+y^2+4x+4y-17=0$ and K is the line $4x+3y=12$.

(i) Show that the line K does not intersect S.

(ii) Find the co-ordinates of the point on S that is closest to K.

SOLUTION

1 (a)

The centre o is the mid-point of $[ab]$.

Mid-point $=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)=\left(\frac{-1+3}{2}, \frac{-3+1}{2}\right)=(1, -1)$

The radius of the circle is half the distance $|ab|$.

$$r=\tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\tfrac{1}{2}\sqrt{(3+1)^2+(1+3)^2}=\tfrac{1}{2}\sqrt{32}=2\sqrt{2}$$

Circle C with centre (h, k), radius r. $(x-h)^2+(y-k)^2=r^2$ 2

C: $(x-1)^2+(y+1)^2=(2\sqrt{2})^2 \Rightarrow (x-1)^2+(y+1)^2=8$. This answer is fine. However, if you decide to expand the equation you will get: $x^2+y^2-2x+2y-6=0$

1 (b) (i)

The radius of the circle is the perpendicular distance from the centre to the tangent.

Centre $(5, -1)$, L: $3x-4y+11=0$

$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ 8

$$d=\frac{|3(5)-4(-1)+11|}{\sqrt{3^2+(-4)^2}}=\frac{|15+4+11|}{\sqrt{25}}=\frac{30}{5}=6$$

1 (b) (ii)

The perpendicular distance of the centre to this line is the radius (6 units).

Centre $(5, -1)$, L: $x + py + 1 = 0$, $d = r = 6$

$$\therefore 6 = \frac{|5 + p(-1) + 1|}{\sqrt{1^2 + p^2}} \Rightarrow 6\sqrt{p^2 + 1} = |6 - p|$$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$ 8

$\Rightarrow 36(p^2 + 1) = 36 - 12p + p^2$ [Square both sides]

$\Rightarrow 36p^2 + 36 = 36 - 12p + p^2 \Rightarrow 35p^2 + 12p = 0$

$\Rightarrow p(35p + 12) = 0 \Rightarrow p = 0, -\frac{12}{35}$

1 (c) (i)

To show that the line K does not intersect the circle S can be done in two ways:

Method 1: Solve K and S simultaneously and show it has no real solutions.

Method 2: Show that the perpendicular distance from the centre of the circle to the line K is greater than the radius of the circle. [This is a better method.]

Method 1: K: $4x + 3y = 12 \Rightarrow x = \frac{12 - 3y}{4}$

$$S:\ x^2 + y^2 + 4x + 4y - 17 = 0 \Rightarrow \left(\frac{12 - 3y}{4}\right)^2 + y^2 + 4\left(\frac{12 - 3y}{4}\right) + 4y - 17 = 0$$

$$\Rightarrow \left(\frac{144 - 72y + 9y^2}{16}\right) + y^2 + 12 - 3y + 4y - 17 = 0$$

$\Rightarrow 144 - 72y + 9y^2 + 16y^2 + 192 - 48y + 64y - 272 = 0$

$\Rightarrow 25y^2 - 56y + 64 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ 4

Remember: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

$a = 25,\ b = -56,\ c = 64$

$b^2 - 4ac = (-56)^2 - 4(25)(64) = 3136 - 6400 = -3264 < 0$

Therefore, there are no real solutions and so K and S do not intersect.

Method 2:

S: $x^2 + y^2 + 4x + 4y - 17 = 0$

Centre $(-2, -2)$, $r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 4 + 17} = 5$

$$d = \frac{|4(-2) + 3(-2) - 12|}{\sqrt{4^2 + 3^2}} = \frac{|-26|}{5} = \frac{26}{5} > 5$$

Circle C centre $(-g, -f)$, radius r.

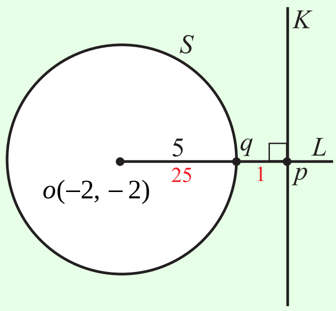
$x^2 + y^2 + 2gx + 2fy + c = 0$ 3

$r = \sqrt{g^2 + f^2 - c}$ 4

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$ 8

1 (c) (ii)

You need to find the co-ordinates of point q. There are two methods to do this. They both involve finding the equation of L, which is perpendicular to K and contains $o(-2,-2)$.



Equation of L: Point $(-2,-2)$, $m=\frac{3}{4}$

L: $3x-4y+k=0$

$(-2,-2)\in L \Rightarrow 3(-2)-4(-2)+k=0 \Rightarrow -6+8+k=0 \Rightarrow k=-2$

Equation of L: $3x-4y-2=0$

Method 1: q divides the line op in the ratio 25:1. [The distance op is $\frac{26}{25}$ whereas the distance oq is $5=\frac{25}{25}$.] You need to find p by solving lines K and L simultaneously.

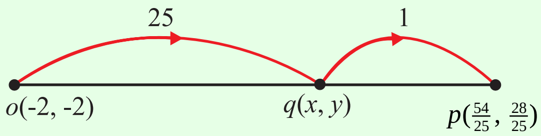
$$\begin{aligned} 4x+3y&=12\,(\times 4)\\ 3x-4y&=2\,(\times 3)\end{aligned}$$

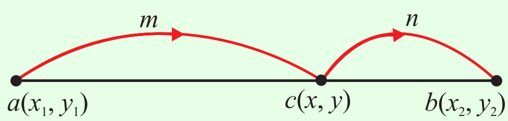
$$\begin{aligned} 16x+12y&=48\\ 9x-12y&=6\\ \hline 25x\qquad &=54 \Rightarrow x=\tfrac{54}{25}\end{aligned}$$

$$4x+3y=12 \Rightarrow y=\frac{12-4x}{3}=\frac{12-4(\frac{54}{25})}{3}=\frac{28}{25}$$

Therefore, the co-ordinates of $p(\frac{54}{25},\frac{28}{25})$.

$$x=\frac{mx_2+nx_1}{m+n},\; y=\frac{my_2+ny_1}{m+n} \quad \ldots\ldots\; \textbf{5}$$





$$x=\frac{25(\frac{54}{25})+1(-2)}{25+1}=\frac{54-2}{26}=\frac{52}{26}=2 \text{ and } y=\frac{25(\frac{28}{25})+1(-2)}{25+1}=\frac{28-2}{26}=\frac{26}{26}=1$$

Therefore, the co-ordinates of $q(2, 1)$.

Method 2: Intersect line L with circle S by solving simultaneously. There will be two solutions. q is the point closer to p.

$$L: 3x-4y=2 \Rightarrow x=\frac{4y+2}{3}$$

$$S:\; x^2+y^2+4x+4y-17=0 \Rightarrow \left(\frac{4y+2}{3}\right)^2+y^2+4\left(\frac{4y+2}{3}\right)+4y-17=0$$

$$\Rightarrow \left(\frac{16y^2+16y+4}{9}\right)+y^2+\left(\frac{16y+8}{3}\right)+4y-17=0$$

$$\Rightarrow 16y^2+16y+4+9y^2+48y+24+36y-153=0$$

$$\Rightarrow 25y^2+100y-125=0 \Rightarrow y^2+4y-5=0$$

$$\Rightarrow (y+5)(y-1)=0 \Rightarrow y=-5,\,1 \Rightarrow x=-6,\,2$$

Therefore, the points of intersection are: $(-6,-5)$, $(2, 1)$

The answer is (2, 1) as it is closer to the line. You can check by using the perpendicular distance formula.