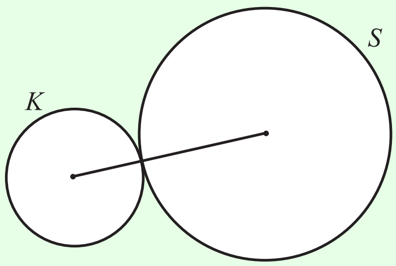
CIRCLE (Q 1, PAPER 2)

2005

1 (a) Circles S and K touch externally. Circle S has entre (8, 5) and radius 6. Circle K has centre $(2, -3)$. Calculate the radius of K.



1 (b) (i) Prove that the equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = r^2$.

(ii) Hence, or otherwise, find the two values of b such that the line $5x + by = 169$ is a tangent to the circle $x^2 + y^2 = 169$.

1 (c) A circle passes through the points (7, 2) and (7, 10). The line $x = -1$ is a tangent to the circle. Find the equation of the circle.

SOLUTION

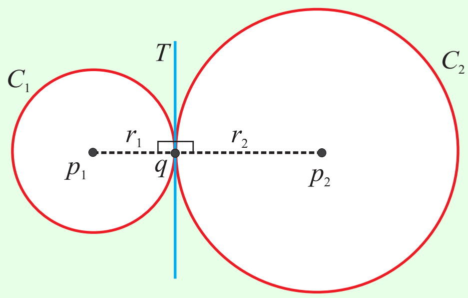
1 (a)

EXTERNAL TOUCH $|p_1p_2| = r_1 + r_2$

$$|p_1p_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(8-2)^2+(5+3)^2}$$

$$= \sqrt{36+64} = 10$$

$$|p_1p_2| = r_1 + r_2 \Rightarrow 10 = 6 + r_2 \Rightarrow r_2 = 4$$



1 (b) (i)

THE TANGENT THEOREM

STATEMENT: Prove that $xx_1 + yy_1 = r^2$ is the equation of the tangent to the circle $x^2 + y^2 = r^2$ at (x_1, y_1).

PROOF

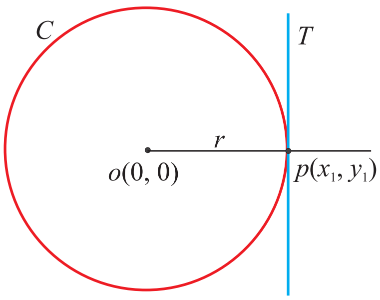
Slope of $op = \dfrac{y_1}{x_1}$

$\therefore$ Slope of $T = -\dfrac{x_1}{y_1}$

$\therefore$ Equation of T: $xx_1 + yy_1 + k = 0$

$(x_1, y_1) \in T \Rightarrow x_1^2 + y_1^2 + k = 0 \Rightarrow k = -x_1^2 - y_1^2 = -r^2$ since $(x_1, y_1) \in S$

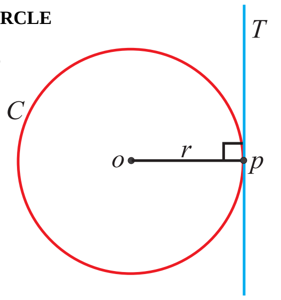
$\therefore$ T: $xx_1 + yy_1 = r^2$



1 (b) (ii)

SOME POINTS YOU NEED TO KNOW ABOUT A TANGENT TO A CIRCLE

1. A tangent T intersects a circle C at one point only, the point of contact p.
2. The perpendicular distance from the centre of the circle C to the tangent T equals the radius r.
3. The tangent T is perpendicular to the line joining the centre o to the point of contact p.



Equation of tangent T: $xx_1 + yy_1 = r^2$ 5

Method 1: Use point No. 2 above.

Circle: $x^2 + y^2 = 169$

Centre (0, 0), $r = 13$

T: $5x + by - 169 = 0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$ 8

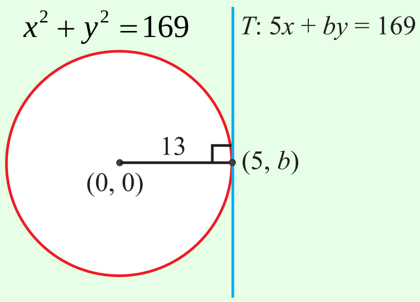
$$\therefore 13 = \frac{|5(0) + b(0) - 169|}{\sqrt{b^2 + 25}} \Rightarrow 13\sqrt{b^2 + 25} = 169$$

$$\Rightarrow \sqrt{b^2 + 25} = 13 \Rightarrow b^2 + 25 = 169$$

$$\Rightarrow b^2 = 144 \Rightarrow b = \pm 12$$

Method 2: Use the equation of tangent formula.

Equation of tangent T: $xx_1 + yy_1 = r^2$ 5



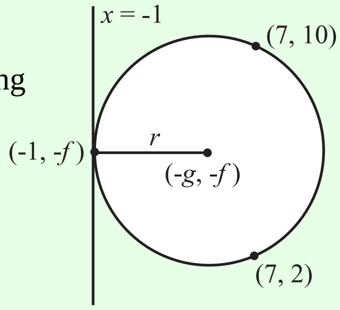
T: $5x + by = 169$
Using the formula, you can see the point of contact is (5, b).
Substitute this point into the circle and solve for b.

$$x^2 + y^2 = 169 \Rightarrow 25 + b^2 = 169 \Rightarrow b^2 = 144 \Rightarrow b = \pm 12$$

1 (c)

You need to find the values of *g*, *f* and *c* and substitute them into the equation

$x^2 + y^2 + 2gx + 2fy + c = 0.$



You are given two points on the circle so you can substitute them into the general equation of the circle. Sketching the circle and tangent reveals more information. You can see that the radius is the distance from the centre $(-g, -f)$ to the point of intersection $(-1, -f)$.

Equation 1: (7, 2) is on the circle.

$49 + 4 + 14g + 4f + c = 0 \Rightarrow 14g + 4f + c = -53 \ldots(\mathbf{1})$

Equation 2: (7, 10) is on the circle.

$49 + 100 + 14g + 20f + c = 0 \Rightarrow 14g + 20f + c = -149 \ldots(\mathbf{2})$

Equation 3: The radius is the distance from the centre $(-g, -f)$ to the point of intersection $(-1, -f)$.

$r = \sqrt{(-g+1)^2 + 0^2} \Rightarrow \sqrt{g^2 + f^2 - c} = \sqrt{(1-g)^2}$

$\Rightarrow g^2 + f^2 - c = (1-g)^2 \Rightarrow g^2 + f^2 - c = 1 - 2g + g^2 \Rightarrow 2g + f^2 - c = 1 \ldots(\mathbf{3})$

Now look at all the equations to find *g*, *f* and *c*.

Combine equations (1) and (2):

$$\begin{aligned} 14g + 20f + c &= -149 \ldots.(2) \\ 14g + 4f + c &= -53 \ldots\ldots.(1)\times(-1) \end{aligned}$$

$\rightarrow$

$$\begin{aligned} 14g + 20f + c &= -149 \\ -14g - 4f - c &= +53 \\ \hline 16f &= -96 \Rightarrow f = -6 \end{aligned}$$

Substitute this value for *f* into equation **3** and equation **1** and solve these equations simultaneously to find *g* and *c*.

Equation **1**: $14g + 4f + c = -53 \Rightarrow 14g - 24 + c = -53 \Rightarrow 14g + c = -29$

Equation **3**: $2g + f^2 - c = 1 \Rightarrow 2g + 36 - c = 1 \Rightarrow 2g - c = -35$

$$\begin{aligned} 14g + c &= -29 \\ 2g - c &= -35 \\ \hline 16g &= -64 \Rightarrow g = -4 \end{aligned}$$

Substitute the values for *f* and *g* into equation **1**:

$14g + 4f + c = -53 \Rightarrow 14(-4) + 4(-6) + c = -53 \Rightarrow c = 27$

Therefore, $g = -4,\ f = -6,\ c = 27.$

Equation of the circle: $x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow x^2 + y^2 - 8x - 12y + 27 = 0$