CIRCLE (Q 1, PAPER 2)

2003

1 (a) For all values of $t \in \mathbf{R}$, the point $\left(\frac{3-3t^2}{1+t^2}, \frac{6t}{1+t^2} \right)$ lies on the circle $x^2 + y^2 = r^2$.

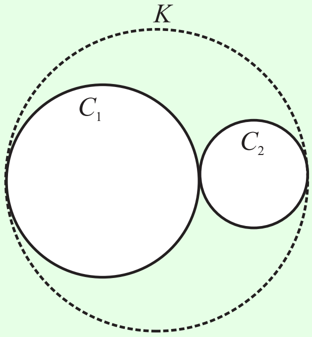
Find r, the radius of the circle.

1 (b) C_1: $x^2 + y^2 + 2x - 2y - 23 = 0$ and

C_2: $x^2 + y^2 - 14x - 2y + 41 = 0$ are two circles.

(i) Prove that C_1 and C_2 touch externally.

(ii) K is a third circle. Both C_1 and C_2 touch K internally. Find the equation of K.



1 (c) The line $ax + by = 0$ is a tangent to the circle $x^2 + y^2 - 12x + 6y + 9 = 0$ where $a, b \in \mathbf{R}$ and $b \neq 0$.

(i) Show that $\frac{a}{b} = -\frac{3}{4}$.

(ii) Hence, or otherwise, find the co-ordinates of the point of contact.

SOLUTION

1 (a)

As the point lies on the circle, you can substitute it into the equation of the circle.

$$x^2 + y^2 = r^2 \Rightarrow \left(\frac{3-3t^2}{1+t^2} \right)^2 + \left(\frac{6t}{1+t^2} \right)^2 = r^2$$

$$\Rightarrow \frac{9 - 18t^2 + 9t^4 + 36t^2}{(1+t^2)^2} = r^2 \Rightarrow \frac{9t^4 + 18t^2 + 9}{(1+t^2)^2} = r^2$$

$$\Rightarrow \frac{9(t^4 + 2t^2 + 1)}{(1+t^2)^2} = r^2 \Rightarrow \frac{9(t^2+1)^2}{(1+t^2)^2} = r^2 \Rightarrow 9 = r^2$$

$$\therefore r = 3$$

1 (b) (i)

Circle C centre $(-g, -f)$, radius r.

$x^2+y^2+2gx+2fy+c=0$ 3

$r=\sqrt{g^2+f^2-c}$ 4

C_1: $x^2+y^2+2x-2y-23=0$

Centre $p_1(-1, 1)$, $r_1=\sqrt{(-1)^2+(1)^2+23}=\sqrt{25}=5$

C_2: $x^2+y^2-14x-2y+41=0$

Centre $p_2(7, 1)$, $r_2=\sqrt{(7)^2+(1)^2-41}=\sqrt{9}=3$

EXTERNAL TOUCH $|p_1p_2|=r_1+r_2$

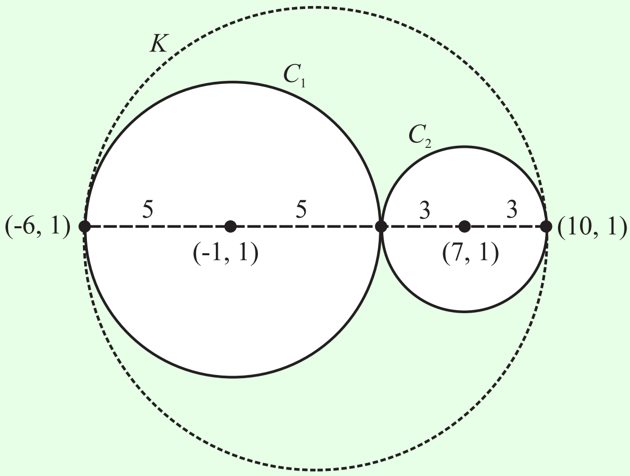
$|p_1p_2|=\sqrt{(-1-7)^2+(1-1)^2}=8$

$r_1+r_2=5+3=8$

Therefore, the two circles touch externally.

1 (b) (ii)

Copy the diagram of the circles. Notice that the diameters are parallel to the X-axis. They coincide with the line y = 1. You can therefore easily find the co-ordinates of the endpoints of the diameter of K.



Centre of K: $\left(\frac{-6+10}{2}, \frac{1+1}{2}\right)=(2, 1)$

Radius of K: r = 8

Circle C with centre (h, k), radius r.

$(x-h)^2+(y-k)^2=r^2$ 2

Equation of K: $(x-2)^2+(y-1)^2=8$

1 (c) (i)

The perpendicular distance from the centre of the circle to the tangent equals the radius of the circle.

Circle: $x^2+y^2-12x+6y+9=0$

Centre $(6,\,-3)$, $r=\sqrt{(6)^2+(-3)^2-9}=\sqrt{36+9-9}=6$

T: $ax+by=0$

$\therefore 6=\dfrac{|a(6)+b(-3)|}{\sqrt{a^2+b^2}} \Rightarrow 6\sqrt{a^2+b^2}=|6a-3b|$ $\quad d=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ 8

$\Rightarrow 2\sqrt{a^2+b^2}=|2a-b|$ [Square both sides.]

$\Rightarrow 4a^2+4b^2=4a^2-4ab+b^2 \Rightarrow 3b^2=-4ab$

$\Rightarrow 3b=-4a \Rightarrow \frac{a}{b}=-\frac{3}{4}$

1 (c) (ii)

$ax+by=0 \Rightarrow x=-\frac{b}{a}y=\frac{4}{3}y$ [Using the previous result]

Substitute this value of x into the circle equation.

$x^2+y^2-12x+6y+9=0 \Rightarrow (\frac{4}{3}y)^2+y^2-12(\frac{4}{3}y)+6y+9=0$

$\Rightarrow \frac{16}{9}y^2+y^2-16y+6y+9=0 \Rightarrow 16y^2+9y^2-90y+81=0$

$\Rightarrow 25y^2-90y+81=0 \Rightarrow (5y-9)(5y-9)=0 \Rightarrow y=\frac{9}{5}$

$x=\frac{4}{3}y=\frac{4}{3}(\frac{9}{5})=\frac{12}{5}$

Ans: $(\frac{12}{5},\,\frac{9}{5})$