CIRCLE (Q 1, PAPER 2)

2002

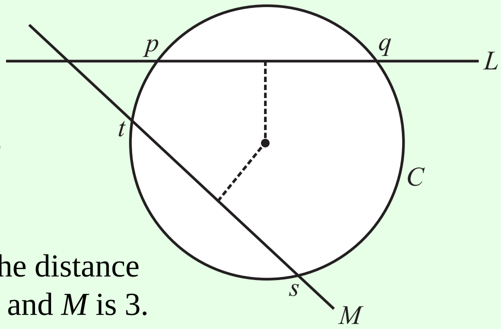
1 (a) The following parametric equations define a circle: $x = 4 + 3\cos\theta$, $y = -2 + 3\sin\theta$, where $\theta \in \mathbf{R}$. What is the Cartesian equation of the circle?

1 (b) The points $a(-2, 4)$, $b(0, -10)$ and $c(6, -2)$ are the vertices of a triangle.

(i) Verify the the triangle is right-angled at c.

(ii) Hence, or otherwise, find the equation of the circle that passes through the points a, b and c.

1 (c) The circle C has equation $x^2 + y^2 - 4x + 6y - 12 = 0$. L intersects C at the points p and q. M intersects C at the points t and s. $|pq| = |ts| = 8$.



(i) Find the radius of C and hence show that the distance from the centre of C to each of the lines L and M is 3.

(ii) Given that L and M intersect at the point $(-4, 0)$, find the equations of L and M.

SOLUTION

1 (a)

> **STEPS**
> **1**. Isolate the trig functions.
> **2**. Square both sides.
> **3**. Add.
> **4**. Put $\cos^2 t + \sin^2 t = 1$.

Parametric Equations: $x = 4 + 3\cos\theta$, $y = -2 + 3\sin\theta$

$$x - 4 = 3\cos\theta \Rightarrow (x-4)^2 = 9\cos^2\theta$$

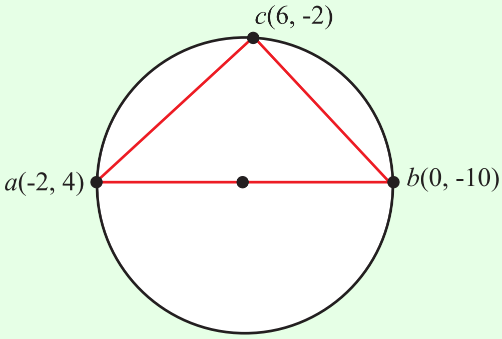
$$y + 2 = 3\sin\theta \Rightarrow (y+2)^2 = 9\sin^2\theta$$

$$(x-4)^2 + (y+2)^2 = 9(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow (x-4)^2 + (y+2)^2 = 9$$

1 (b) (i)

To show $ac \perp bc$



$m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$ 2

Slope of ac: $m_1 = \dfrac{4+2}{-2-6} = \dfrac{6}{-8} = -\dfrac{3}{4}$

Slope of bc: $m_2 = \dfrac{-10+2}{0-6} = \dfrac{-8}{-6} = \dfrac{4}{3}$

Two lines are perpendicular if the product of their slopes is -1.
$K \perp L \Leftrightarrow m_1 \times m_2 = -1.$

$m_1 \times m_2 = (-\tfrac{3}{4})(\tfrac{4}{3}) = -1 \Rightarrow ac \perp bc$

1 (b) (ii)

The best way to do this question is to find the centre and radius of the circle from the points given. This is easily done if you remember a theorem from your Junior Cert, i.e. the angle standing on the diameter of a circle is a right-angle. As $\angle acb$ is a right angle, it follows that [*ab*] is the diameter of the circle.
Therefore, the centre is the midpoint of [ab].

Centre: $\left(\dfrac{-2+0}{2}, \dfrac{4-10}{2}\right) = (-1, -3)$

The radius is the distance from the centre to any point, say *a*.

$r = \sqrt{(-2+1)^2 + (4+3)^2} = \sqrt{1+49} = \sqrt{50}$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ 1

Equation of circle: Centre $(-1, -3)$, $r = \sqrt{50}$

$(x+1)^2 + (y+3)^2 = 50$

This answer is fine. If you multiply this equation out you get: $x^2 + y^2 + 2x + 6y - 40 = 0$

Circle *C* with centre (*h*, *k*), radius *r*.

$(x-h)^2 + (y-k)^2 = r^2$ 2

1 (c) (i)

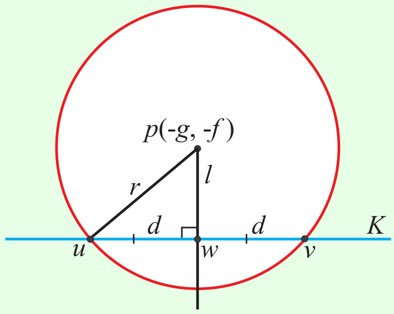
Circle C centre $(-g,-f)$, radius r.

$x^2+y^2+2gx+2fy+c=0$ **3**

$r=\sqrt{g^2+f^2-c}$ **4**

Some properties of chords

1. The line K intersects the circle at points u and v.
2. $[uv]$ is a chord.
3. The mid-point of the chord $[uv]$ is w.
4. The line from the centre of the circle to w is perpendicular to the chord.
5. You can apply Pythagoras by completing a right-angled triangle.
6. The perpendicular distance of p to K is the distance l. Obviously, $l < r$.

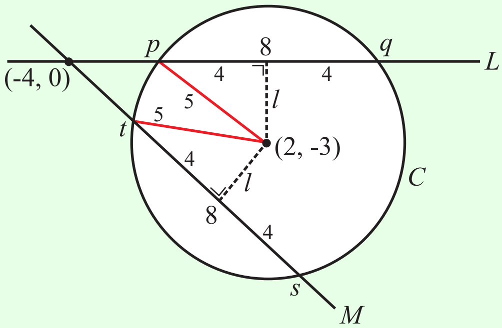


Circle C: $x^2+y^2-4x+6y-12=0$

Centre $(2,-3)$, $r=\sqrt{4+9+12}=5$

Apply Pythagoras to the right-angled triangles to show the distance l is 3.

$\therefore 4^2+l^2=5^2 \Rightarrow l^2=25-16=9 \Rightarrow l=3$



1 (c) (ii)

Equations of L and M: Point $(-4, 0)$, Slope $=+\frac{m}{1}$

$\Rightarrow mx-y+k=0$

$\Rightarrow m(-4)-(0)+k=0 \Rightarrow k=4m$

$\Rightarrow mx-y+4m=0$ .....(**1**)

You know that the perpendicular distance from the centre to L and M is 3.

$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ **8**

$3=\frac{|m(2)-(-3)+4m|}{\sqrt{m^2+1}} \Rightarrow 3\sqrt{m^2+1}=|6m+3| \Rightarrow \sqrt{m^2+1}=|2m+1|$

$\Rightarrow m^2+1=4m^2+4m+1 \Rightarrow 3m^2+4m=0$

$\Rightarrow m(3m+4)=0 \Rightarrow m=0, -\frac{4}{3}$

Substitute these values of m into equation **1** to give the two equations L and M.

$m=0 \Rightarrow y=0$

$m=-\frac{4}{3} \Rightarrow -\frac{4}{3}x-y+4(-\frac{4}{3})=0 \Rightarrow 4x+3y+16=0$