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ALGEBRA (Q 1 & 2, PAPER 1)

SOLUTIONS NO. 7: FUNCTIONS

2006

2 (c) $f(x)=1-b^{2x}$ and $g(x)=b^{1+2x}$, where b is a positive real number. Find, in terms of b, the value of x for which $f(x)=g(x)$.

SOLUTION

$f(x)=g(x)\Rightarrow 1-b^{2x}=b^{1+2x}$

$\Rightarrow 1=b^{1+2x}+b^{2x}\Rightarrow 1=b^{2x}(b+1)$

$\Rightarrow \dfrac{1}{(b+1)}=b^{2x}\Rightarrow \log_{10}\left(\dfrac{1}{(b+1)}\right)=\log_{10}b^{2x}$

$\Rightarrow -\log_{10}(b+1)=2x\log_{10}b$

$\Rightarrow -\dfrac{\log_{10}(b+1)}{2\log_{10}b}=x$

OR take the log to base b.

$\dfrac{1}{(b+1)}=b^{2x}\Rightarrow \log_b\left(\dfrac{1}{b+1}\right)=\log_b b^{2x}$

$\Rightarrow -\log_b(b+1)=2x\log_b b$

$\Rightarrow -\tfrac{1}{2}\log_b(b+1)=x$

$\Rightarrow -\log_b\sqrt{b+1}=x$

2005

2 (c) Let $f(x)=\dfrac{x^2+k^2}{mx}$, where k and m are constants and $m\neq 0$.

(i) Show that $f(km)=f\left(\dfrac{k}{m}\right)$.

(ii) a and b are real numbers such that $a\neq 0,\ b\neq 0$ and $a\neq b$. Show that if $f(a)=f(b)$, then $ab=k^2$.

SOLUTION

2 (c) (i)

$$f(km)=\frac{(km)^2+k^2}{m(km)}=\frac{k^2m^2+k^2}{km^2}=\frac{k^2(m^2+1)}{km^2}=\frac{k(m^2+1)}{m^2}$$

$$f\left(\frac{k}{m}\right)=\frac{(\frac{k}{m})^2+k^2}{m(\frac{k}{m})}=\frac{\frac{k^2}{m^2}+k^2}{k}\times\frac{m^2}{m^2}=\frac{k^2+k^2m^2}{km^2}=\frac{k^2(m^2+1)}{km^2}=\frac{k(m^2+1)}{m^2}$$

2 (c) (ii)

$f(a)=\dfrac{a^2+k^2}{ma}$ and $f(b)=\dfrac{b^2+k^2}{mb}$

If $f(a)=f(b)\Rightarrow \dfrac{a^2+k^2}{ma}=\dfrac{b^2+k^2}{mb}\Rightarrow b(a^2+k^2)=a(b^2+k^2)$

$\Rightarrow ba^2+bk^2=ab^2+ak^2\Rightarrow ba^2-ab^2=ak^2-bk^2$

$\Rightarrow ab(a-b)=k^2(a-b)\Rightarrow ab=k^2$

2004

2 (c) (i) $f(x) = 2x + 1$, for $x \in \mathbf{R}$. Show that there exists a real number k such that for all x, $f(x + f(x)) = kf(x)$.

SOLUTION

$f(x + f(x)) = kf(x) \Rightarrow f(x + 2x + 1) = k(2x + 1)$

$\Rightarrow f(3x + 1) = k(2x + 1) \Rightarrow 2(3x + 1) + 1 = k(2x + 1)$

$\Rightarrow 6x + 3 = k(2x + 1) \Rightarrow 3(2x + 1) = k(2x + 1)$

$\Rightarrow k = 3$

2002

2 (b) (ii) Let $g(x) = x^n + 3$, for all $x \in \mathbf{R}$, where $n \in \mathbf{N}$. Show that if n is odd then $g(x) + g(-x)$ is constant.

SOLUTION

A negative number raised to an even power gives a positive answer. However, a negative number raised to an odd power gives a negative answer.

$g(x) = x^n + 3$

$g(-x) = (-x)^n + 3 = -x^n + 3$, as n is an odd power.

$\therefore g(x) + g(-x) = x^n + 3 - x^n + 3 = 6$ [i.e. a constant]