# ALGEBRA (Q 1 & 2, PAPER 1)

## SOLUTIONS NO. 4: QUADRATIC EQUATIONS

**2006**

2 (b) (i) Find the range of values of $t \in \mathbf{R}$ for which the quadratic equation
$(2t-1)x^2 + 5tx + 2t = 0$ has real roots.

(ii) Explain why the roots are real when $t$ is an integer.

> **REMEMBER**: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
> If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

### SOLUTION

**2 (b) (i)**

$a = 2t - 1$

$b = 5t$

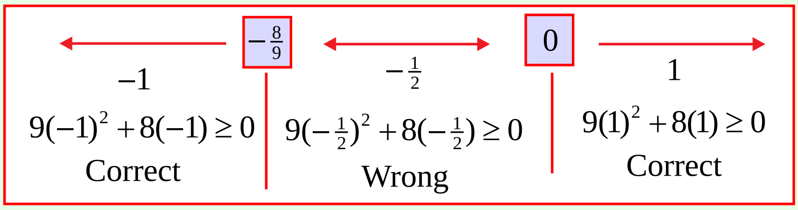
$c = 2t$

$(5t)^2 - 4(2t-1)(2t) \geq 0 \Rightarrow 9t^2 + 8t \geq 0$

Solve $9t^2 + 8t = 0 \Rightarrow t(9t+8) = 0 \Rightarrow t = 0, -\frac{8}{9}$

Roots: $\alpha = -\frac{8}{9}$, $\beta = 0$

Region Test:




Region Test on $\boxed{9t^2 + 8t \geq 0}$ .....**Test Box**

$\therefore t \leq -\frac{8}{9}$, $t \geq 0$

**2 (b) (ii)**

An integer is a whole number. The solutions are unreal for values of $t$ in the range $-\frac{8}{9} \leq t \leq 0$. There are no integers in this range and therefore, the roots are real when $t$ is an integer.

**2004**

2 (b) (ii) The roots of $x^2 + px + q = 0$ are $\alpha$ and $\beta$, where $p, q \in \mathbf{R}$. Find the quadratic equation whose roots are $\alpha^2\beta$ and $\alpha\beta^2$.

2 (c) (ii) Show that for any real values of $a$, $b$ and $h$, the quadratic equation $(x-a)(x-b)-h^2=0$ has real roots.

**SOLUTION**

**2 (b) (ii)**

**OLD QUADRATIC**

$x^2 + px + q = 0$

Roots: $\alpha, \beta$

Sum **S**: $\alpha + \beta = -p$

Product **P**: $\alpha\beta = q$

**NEW QUADRATIC**

$x^2 - \mathbf{S}x + \mathbf{P} = 0$

Roots: $\alpha^2\beta, \alpha\beta^2$

Sum **S**: $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta) = -pq$

Product **P**: $(\alpha^2\beta)(\alpha\beta^2) = (\alpha\beta)^3 = q^3$

Equation: $x^2 + pqx + q^3 = 0$

**2 (c) (ii)**

**REMEMBER**: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

$(x-a)(x-b)-h^2 = 0 \Rightarrow x^2 - (a+b)x + ab - h^2 = 0$

If the roots are real you need to show $(a+b)^2 - 4(ab-h^2) \geq 0$.

$(a+b)^2 - 4(ab-h^2) = a^2 + 2ab + b^2 - 4ab + 4h^2 = a^2 - 2ab + b^2 + 4h^2$

$= (a-b)^2 + 4h^2 \geq 0$ always.

**2003**

1 (c) The real roots of $x^2+10x+c=0$ differ by $2p$ where $c,\ p\in\mathbf{R}$ and $p>0$.

(i) Show that $p^2=25-c$.

(ii) Given that one root is greater than zero and the other root is less than zero, find the range of possible values of $p$.

2 (c)(ii) Given that $x=\alpha$ and $x=\beta$ are the solutions of the quadratic equation $2k^2x^2+2ktx+t^2-3k^2=0$ where $k,\ t\in\mathbf{R}$ and $k\neq 0$, show that $\alpha^2+\beta^2$ is independent of $k$ and $t$.

**SOLUTION**

**1 (c) (i)**

$x^2+10x+c=0$

Roots: $\alpha,\ \alpha+2p$ [The roots differ by $2p$]

Sum **S**: $\alpha+\alpha+2p=-10\Rightarrow\alpha+p=-5$ ... (**1**)

Product **P**: $\alpha(\alpha+2p)=c$ ... (**2**)

From equation **1**: $\alpha=-p-5$

Substituting into equation **2**: $(-p-5)(-p-5+2p)=c\Rightarrow(-p-5)(p-5)=c$

$\Rightarrow 25-p^2=c\Rightarrow p^2=25-c$

**1 (c) (ii)**

Call the roots $\alpha,\ \beta$

$\alpha=-p-5$ [As $p>0$, this is the negative root]

$\beta=\alpha+2p=-p-5+2p=p-5$ [Therefore, this is the positive root. However, it can be seen that it is only positive for values of $p$ greater than 5.]

Answer: $p>5$

**2 (c) (ii)**

$2k^2x^2+2ktx+t^2-3k^2=0$

Roots: $\alpha,\ \beta$

Sum **S**: $\alpha+\beta=-\dfrac{2kt}{2k^2}=-\dfrac{t}{k}$

Product **P**: $\alpha\beta=\dfrac{t^2-3k^2}{2k^2}$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{t}{k}\right)^2-2\left(\frac{t^2-3k^2}{2k^2}\right)$$

$$=\frac{t^2}{k^2}-\frac{t^2-3k^2}{k^2}=\frac{3k^2}{k^2}=3$$

**2002**

1 (c) $(p+r-t)x^2+2rx+(t+r-p)=0$ is a quadratic equation, where $p$, $r$, and $t$ are integers. Show that

(i) the roots are rational
(ii) one of the roots is an integer.

2 (c) (i) Show that if the roots of $x^2+bx+c=0$ differ by 1, then $b^2-4c=1$.

(ii) The roots of the equation $x^2+(4k-5)x+k=0$ are consecutive integers. Using the result from part (i), or otherwise, find the value of $k$ and the roots of the equation.

**SOLUTION**

**1 (c)**

$(p+r-t)x^2+2rx+(t+r-p)=0$

$a=(p+r-t)=[r+(p-t)]$
$b=2r$
$c=(t+r-p)=[r-(p-t)]$

$$x=\frac{-2r\pm\sqrt{4r^2-4[r+(p-t)][r-(p-t)]}}{2(p+r-t)}$$

$$=\frac{-2r\pm\sqrt{4r^2-4r^2+4(p-t)^2}}{2(p+r-t)}=\frac{-2r\pm\sqrt{4(p-t)^2}}{2(p+r-t)}=\frac{-2r\pm2(p-t)}{2(p+r-t)}$$

$$x=\frac{-2r+2p-2t}{2(p+r-t)}=\frac{2(p-r-t)}{2(p+r-t)}=\frac{p-r-t}{p+r-t}$$

or $x=\dfrac{-2r-2p+2t}{2(p+r-t)}=\dfrac{-2(p+r-t)}{2(p+r-t)}=-1$

**2 (c) (i)**

$x^2+bx+c=0$

Roots: $\alpha,\ \alpha+1$

Sum **S**: $\alpha+\alpha+1=-b\Rightarrow 2\alpha+1=-b$...(**1**)

Product **P**: $\alpha(\alpha+1)=c$...(**2**)

From equation **1**: $2\alpha+1=-b\Rightarrow\alpha=-\dfrac{b+1}{2}$

Substituting into equation **2**:

$$\alpha(\alpha+1)=c\Rightarrow\left(-\frac{b+1}{2}\right)\left(-\frac{b+1}{2}+1\right)=c\Rightarrow\left(\frac{-b-1}{2}\right)\left(\frac{-b+1}{2}\right)=c$$

$\Rightarrow b^2-1=4c\Rightarrow b^2-4c=1$

**2 (c) (ii)**

$x^2+(4k-5)x+k=0$

As the roots are consecutive integers $\Rightarrow b^2-4c=1$.

$\therefore (4k-5)^2-4k=1 \Rightarrow 16k^2-40k+25-4k-1=0$

$\Rightarrow 16k^2-44k+24=0 \Rightarrow 4k^2-11k+6=0$

$\Rightarrow (4k-3)(k-2)=0 \Rightarrow k=\frac{3}{4},\ 2$

Which value of $k$ gives integer roots?

$k=\frac{3}{4} \Rightarrow x^2+(4(\frac{3}{4})-5)x+(\frac{3}{4})=0 \Rightarrow x^2-2x+\frac{3}{4}=0$

$\Rightarrow 4x^2-8x+3=0 \Rightarrow (2x-3)(2x-1)=0 \Rightarrow x=\frac{1}{2},\ \frac{3}{2}$ [Not integer roots]

$k=2 \Rightarrow x^2+(4(2)-5)x+2=0 \Rightarrow x^2+3x+2=0$

$\Rightarrow (x+2)(x+1)=0 \Rightarrow x=-2,\ -1$ [Integer roots]

**Answer**: $k=2$; Roots: $-2,\ -1$

**2001**

2 (c) $\alpha$ and $\beta$ are real numbers such that $\alpha+\beta=-7$ and $\alpha\beta=11$.

(i) Show that $\alpha^2+\beta^2=27$.

(ii) Find a quadratic equation with roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ and write your answer in the form

$px^2+qx+r=0$ where $p,\ q,\ r\in\mathbf{Z}$.

**Solution**

**2 (c) (i)**

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-7)^2-2(11)=49-22=27$

**2 (c) (ii)**

Roots: $\dfrac{\alpha}{\beta},\ \dfrac{\beta}{\alpha}$

Sum **S**: $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{27}{11}$

Product **P**: $\left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right)=1$

Using $x^2-\mathbf{S}x+\mathbf{P}=0 \Rightarrow x^2-\frac{27}{11}x+1=0 \Rightarrow 11x^2-27x+11=0$