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ALGEBRA (Q 1 & 2, PAPER 1)

SOLUTIONS NO. 2: IDENTITIES

2006

1 (c) $x^2 - t$ is a factor of $x^3 - px^2 - qx + r$.

(i) Show that $pq = r$.

(ii) Express the roots of $x^3 - px^2 - qx + r = 0$ in terms of p and q.

SOLUTION

METHOD 1: DIVISION PROCESS

1 (c) (i)

$$
\begin{array}{r|l}
 & x - p \\
\hline
x^2 - t & x^3 - px^2 - qx + r \\
 & \mp x^3 \qquad\quad \pm tx \\
\hline
 & \quad -px^2 + (t-q)x + r \\
 & \quad \mp px^2 \qquad\quad \pm pt \\
\hline
 & \qquad\qquad (t-q)x + (r - pt)
\end{array}
$$

The remainder has to be zero, i.e. $0x + 0$.

$\therefore t - q = 0 \Rightarrow t = q$ and $r - pt = 0 \Rightarrow r = pt$

$\therefore r = pq$

1 (c) (ii)

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - p) = 0$

$\Rightarrow x = \pm\sqrt{t},\ p \Rightarrow x = \pm\sqrt{q},\ p$

METHOD 2: LINING UP

1 (c) (i)

A cubic is a quadratic multiplied by a linear. The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear. Also, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear.

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - \frac{r}{t})$

$\Rightarrow (x^3 - px^2 - qx + r) = x^3 - \frac{r}{t}x^2 - tx + r$

Lining up the coefficients: $\Rightarrow p = \frac{r}{t}$ and $q = t$.

$\therefore p = \frac{r}{q} \Rightarrow r = pq$

1 (c) (ii)

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - p) = 0$

$\Rightarrow x = \pm\sqrt{t},\ p \Rightarrow x = \pm\sqrt{q},\ p$

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2005

1 (c) $(x-p)^2$ is a factor of x^3+qx+r. Show that $27r^2+4q^3=0$. Express the roots of $3x^2+q=0$ in terms of p.

SOLUTION

METHOD 1: DIVISION PROCESS

$(x-p)^2 = x^2-2px+p^2$

$$\begin{array}{r|l} & x+2p \\ \hline x^2-2px+p^2 & x^3+0x^2+qx+r \\ & \mp x^3 \pm 2px^2 \mp p^2x \\ \hline & \quad 2px^2+(q-p^2)x+r \\ & \quad \mp 2px^2 \pm 4p^2x \mp 2p^3 \\ \hline & \quad (3p^2+q)x+(r-2p^3) \end{array}$$

The remainder has to be zero, i.e. $0x + 0$.

$\therefore 3p^2+q=0 \Rightarrow q=-3p^2$ and $r-2p^3=0 \Rightarrow r=2p^3$

$27r^2+4q^3=27(2p^3)^2+4(-3p^2)^3=27(4p^6)+4(-27p^6)$

$=108p^6-108p^6=0$

$$3x^2+q=0 \Rightarrow x=\sqrt{-\frac{q}{3}}=\sqrt{-\frac{-3p^2}{3}}=\sqrt{p^2}=\pm p$$

METHOD 2: LINING UP

$(x-p)^2 = x^2-2px+p^2$

$x^3+0x^2+qx+r=(x^2-2px+p^2)(x+\frac{r}{p^2})$

$x^3+0x^2+qx+r=x^3+(\frac{r}{p^2}-2p)x+(p^2-\frac{2r}{p})x+r$

Lining up: $(\frac{r}{p^2}-2p)=0 \Rightarrow r=2p^3$ and $(p^2-\frac{2r}{p})=q \Rightarrow -3p^2=q$

$27r^2+4q^3=27(2p^3)^2+4(-3p^2)^3=27(4p^6)+4(-27p^6)$

$=108p^6-108p^6=0$

$$3x^2+q=0 \Rightarrow x=\sqrt{-\frac{q}{3}}=\sqrt{-\frac{-3p^2}{3}}=\sqrt{p^2}=\pm p$$

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2003

2 (b) (ii) Given that $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ where $a, b \in \mathbf{R}$, find the value of a and the value of b.

SOLUTION

Lining up is the best way to do this.

$$x^3 - 5x^2 + bx + 9 = (x^2 - ax - 3)(x - 3)$$

$$\Rightarrow x^3 - 5x^2 + bx + 9 = x^3 - (a+3)x^2 + (3a-3)x + 9$$

Lining up: $a + 3 = 5 \Rightarrow a = 2$ and $3a - 3 = b \Rightarrow b = 3$

2001

1 (a) Find the real numbers a and b such that $x^2 + 4x - 6 = (x+a)^2 + b$ for all $x \in \mathbf{R}$.

1 (c) $x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$.

(i) Show that $q = -2p^2$.

(ii) Show that $r = -8p^3$.

(iii) Find the three roots of $x^3 + 3px^2 + 3qx + r = 0$ in terms of p.

SOLUTION

1 (a)

Multiply out the brackets and line up the coefficients.

$$x^2 + 4x - 6 = (x+a)^2 + b \Rightarrow x^2 + 4x - 6 = x^2 + 2ax + a^2 + b$$

Therefore, $2a = 4$ and $a^2 + b = -6 \Rightarrow a = 2,\ b = -10$

1 (c)

The division method only is shown here. Try lining up yourself if that is your favoured method.

$$\begin{array}{r|l}
 & x + 4p \\
\hline
x^2 - px + q & x^3 + 3px^2 + 3qx + r \\
 & \mp x^3 \pm px^2 \mp qx \\
\hline
 & \quad 4px^2 + 2qx + r \\
 & \quad \mp 4px^2 \pm 4p^2x \mp 4pq \\
\hline
 & \qquad (4p^2 + 2q)x + (r - 4pq)
\end{array}$$

The remainder has to be zero, i.e. $0x + 0$.

$$\therefore 4p^2 + 2q = 0 \Rightarrow q = -2p^2$$

and $r = 4pq \Rightarrow r = 4p(-2p^2) = -8p^3$

$$x^3 + 3px^2 + 3qx + r = (x^2 - px + q)(x + 4p)$$

$$\Rightarrow x^3 + 3px^2 + 3qx + r = (x^2 - px - 2p^2)(x + 4p) = (x - 2p)(x + p)(x + 4p)$$

$$\therefore x = -4p,\ -p,\ 2p$$