ALGEBRA (Q 1 & 2, PAPER 1)

SOLUTIONS NO. 1: SOME BASICS

2006

1 (a) Find the real number a such that for all $x \neq 9$, $\dfrac{x-9}{\sqrt{x}-3} = \sqrt{x} + a$.

SOLUTION

Turn $x-9$ in a difference of 2 squares and factorise.

$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x})^2-(3)^2}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = (\sqrt{x}+3)$$

Therefore, $a = 3$.

OR

Cross-multiply: $\dfrac{x-9}{\sqrt{x}-3} = \sqrt{x} + a \Rightarrow x-9 = (\sqrt{x}-3)(\sqrt{x}+a)$

$\Rightarrow x-9 = x + (a-3)\sqrt{x} - 3a$

Lining up coefficents: $\Rightarrow -9 = -3a \Rightarrow a = 3$

2005

1 (b) (i) Express $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}}$ in the form $2^{\frac{p}{q}}$, where $p, q \in \mathbf{Z}$.

(ii) Let $f(x) = ax^3 + bx^2 + cx + d$. Show that $(x-t)$ is a factor of $f(x) - f(t)$.

SOLUTION

1 (b) (i)

$2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} = 4 \times 2^{\frac{1}{4}} = 2^2 \times 2^{\frac{1}{4}} = 2^{\frac{9}{4}}$ [Using power rule No. 1: $a^m \times a^n = a^{m+n}$]

1 (b) (ii)

$f(x) = ax^3 + bx^2 + cx + d$

$f(t) = at^3 + bt^2 + ct + d$

$f(x) - f(t) = a(x^3 - t^3) + b(x^2 - t^2) + c(x-t)$ [Factorise using Formulae 1 and 3.]

$\Rightarrow f(x) - f(t) = a(x-t)(x^2 + xt + t^2) + b(x-t)(x+t) + c(x-t)$

$\Rightarrow f(x) - f(t) = (x-t)\{a(x^2 + xt + t^2) + b(x+t) + c\}$

Therefore, $(x-t)$ is a factor of $f(x) - f(t)$.

2004

1 (a) Express $\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$ in the form $a\sqrt{3}-b$, where a and $b \in \mathbf{N}$.

1 (b) (ii) Show that $\dfrac{3}{1+x^p}+\dfrac{3}{1+x^{-p}}$ simplifies to a constant.

1 (c) (i) Show that $p^3+q^3-(p+q)^3=-3pq(p+q)$.

(ii) Hence, or otherwise, find, in terms of a and b, the three values of x for which

$(a-x)^3+(b-x)^3-(a+b-2x)^3=0$.

SOLUTION

1 (a)

$\dfrac{1-\sqrt{3}}{1+\sqrt{3}}=\dfrac{(1-\sqrt{3})}{(1+\sqrt{3})}\times\dfrac{(1-\sqrt{3})}{(1-\sqrt{3})}$ [Multiply above and below by the conjugate of denominator.]

$=\dfrac{1-\sqrt{3}-\sqrt{3}+3}{-2}=\dfrac{4-2\sqrt{3}}{-2}=\sqrt{3}-2$

1 (b) (ii)

Always deal with a negative power immediately by multiplying above and below by the opposite positive power.

$\therefore \dfrac{3}{1+x^{-p}}\times\dfrac{x^p}{x^p}=\dfrac{3x^p}{x^p+1}$

$\Rightarrow \dfrac{3}{1+x^p}+\dfrac{3}{1+x^{-p}}=\dfrac{3}{1+x^p}+\dfrac{3x^p}{1+x^p}=\dfrac{3+3x^p}{1+x^p}=\dfrac{3(1+x^p)}{1+x^p}=3$

1 (c) (i)

$p^3+q^3-(p+q)^3=p^3+q^3-(p^3+3p^2q+3pq^2+q^3)$ [Using multiplying out brackets on page 1]

$=p^3+q^3-p^3-3p^2q-3pq^2-q^3=-3p^2q-3pq^2$

$=-3pq(p-q)$

1 (c) (ii)

$p \leftrightarrow (a-x),\ q \leftrightarrow (b-x),\ p+q \leftrightarrow (a+b-2x)$

If $p^3+q^3-(p+q)^3=-3pq(p+q)$

$\Rightarrow (a-x)^3+(b-x)^3-(a+b-2x)^3=-3(a-x)(b-x)(a+b-2x)=0$

Set each bracket equal to zero to solve for x.

$\Rightarrow x=a,\ b,\ \frac{1}{2}(a+b)$

2003

1 (a) Express the following as a single fraction in its simplest form: $\frac{6y}{x(x+4y)} - \frac{3}{2x}$.

SOLUTION

$$\frac{6y}{x(x+4y)} - \frac{3}{2x} = \frac{12y - 3(x+4y)}{2x(x+4y)} = \frac{12y - 3x - 12y}{2x(x+4y)}$$

$$= \frac{-3x}{2x(x+4y)} = -\frac{3}{2(x+4y)}$$

2001

2 (b) (ii) Simplify $\left(x^2 + \sqrt{2} + \frac{1}{x^2}\right)\left(x^2 - \sqrt{2} + \frac{1}{x^2}\right)$ and express your answer in the form

$x^n + \frac{1}{x^n}$ where *n* is a whole number.

SOLUTION

2 (b) (ii)

$$\left(x^2 + \sqrt{2} + \frac{1}{x^2}\right)\left(x^2 - \sqrt{2} + \frac{1}{x^2}\right) = x^4 - \sqrt{2}x^2 + 1 + \sqrt{2}x^2 - 2 + \frac{\sqrt{2}}{x^2} + 1 - \frac{\sqrt{2}}{x^2} + \frac{1}{x^4}$$

$$= x^4 + \frac{1}{x^4}$$