ALGEBRA (Q 1 & 2, PAPER 1)

2011

1 **(a)** Simplify fully $\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4}{x^2-1}$.

(b) **(i)** Prove the factor theorem for polynomials of degree 2.

That is, given that $f(x)=ax^2+bx+c$ and k is a number such that $f(k)=0$, prove that $(x-k)$ is a factor of $f(x)$.

(ii) The factor theorem also holds for polynomials of higher degree.
Find the values of n for which $(x+k)$ is a factor of the polynomial

$g(x)=x^n+k^n$, where $k\neq 0$.

(c) $(x-a)^2$ is a factor of $2x^3-5ax^2+8abx-36a$, where $a\neq 0$.
Find the possible values of a and b.

SOLUTION

1 (a)

Factorise denominators first.
Get the lowest common denominator (LCD).

$$\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4}{x^2-1}$$

$$=\frac{x+1}{(x-1)}-\frac{x-1}{(x+1)}-\frac{4}{(x+1)(x-1)}$$

$$=\frac{(x+1)(x+1)-(x-1)(x-1)-4}{(x+1)(x-1)}$$

$$=\frac{x^2+2x+1-x^2+2x-1-4}{(x+1)(x-1)}$$

$$=\frac{4x-4}{(x+1)(x-1)}$$

$$=\frac{4\,\cancel{(x-1)}}{(x+1)\,\cancel{(x-1)}}$$

$$=\frac{4}{x+1}$$

1 (b) (i)

PROOF OF FACTOR THEOREM FOR A CUBIC

$f(x) = ax^3 + bx^2 + cx + d$

$f(k) = ak^3 + bk^2 + ck + d$

$\therefore\ f(x) - f(k) = a(x^3 - k^3) + b(x^2 - k^2) + c(x - k)$

$= (x-k)\{ax^2 + akx + ak^2 + bx + bk + c\} = (x-k)g(x)$

$\therefore\ f(x) = f(k) + (x-k)g(x)$

(i) $f(k) = 0 \Rightarrow f(x) = (x-k)g(x)\ \therefore\ x - k$ is a factor.

(ii) $x - k$ is a factor $\Rightarrow f(k) = 0$.

1 (b) (ii)

If $(x + k)$ is a factor $\Rightarrow x = -k$ is a root.

$\therefore\ g(-k) = 0 \Rightarrow (-k)^n + (k)^n = 0.$

This is only true if n is an odd natural number.

$n = \{1, 3, 5, 7, \ldots\ldots.\}$

1 (c)

A cubic is a quadratic multiplied by a linear.

The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear.

Also, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear.

$$2x^3 - 5ax^2 + 8abx - 36a = (x^2 - 2ax + a^2)(2x - \tfrac{36}{a})$$

$$2x^3 - 5ax^2 + 8abx - 36a = 2x^3 - \tfrac{36}{a}x^2 - 4ax^2 + 72x + 2a^2x - 36a$$

$$2x^3 - 5ax^2 + 8abx - 36a = 2x^3 + (-4a - \tfrac{36}{a})x^2 + (2a^2 + 72) - 36a$$

Line up the coefficients.

$-5a = -4a - \tfrac{36}{a}$

$\tfrac{36}{a} = a$

$36 = a^2$

$\therefore\ a = \pm 6$

$2a^2 + 72 = 8ab$

$a^2 + 36 = 4ab$

$$\therefore\ b = \frac{a^2 + 36}{4a}$$

$$a = 6 \Rightarrow b = \frac{(6)^2 + 36}{4(6)} = \frac{72}{24} = 3$$

$$a = -6 \Rightarrow b = \frac{(-6)^2 + 36}{4(-6)} = \frac{72}{-24} = -3$$

2. **(a)** Solve for x: $|2x-1| \leq 3$, where $x \in \mathbb{R}$.

(b) α and $\frac{1}{\alpha}$ are the roots of the quadratic equation $3kx^2 - 18tx + (2k+3) = 0$, where t and k are constants.

(i) Find the value of k.

(ii) If one of the roots is four times the other, find the possible values of t.

(c) Let $f(x) = \dfrac{1}{x^2 - 6x + a}$, where a is a constant.

(i) Prove that if $a = 13$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

(ii) Prove that if $a = 13$, then $f(x) < 1$ for all $x \in \mathbb{R}$.

(iii) Find the range of values of a such that $0 < f(x) < 1$, for all $x \in \mathbb{R}$.

Solution

2 (a)

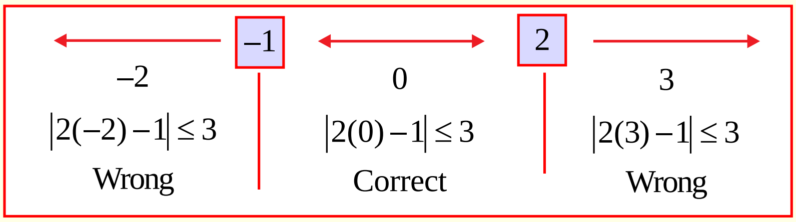
Solve the equality:

$|2x-1| = 3 \Rightarrow 2x - 1 = \pm 3$

$2x - 1 = 3 \Rightarrow x = 2$

$2x - 1 = -3 \Rightarrow x = -1$

Do the region test:



Region Test on $|2x-1| \leq 3$ .....**Test Box**

$\therefore -1 \leq x \leq 3$

2 (b)

$$3kx^2 - 18tx + (2k+3) = 0$$

$$\text{Roots: } \alpha, \frac{1}{\alpha}$$

$$\mathbf{S}: \alpha + \frac{1}{\alpha} = \frac{-18t}{3k} = -\frac{6t}{k}$$

$$\mathbf{P}: \alpha \times \frac{1}{\alpha} = \frac{2k+3}{3k} \Rightarrow 1 = \frac{2k+3}{3k}$$

Sum S: $\alpha + \beta = -\frac{b}{a} = \frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta = \frac{c}{a} = \frac{3^{\text{rd.}}}{1^{\text{st.}}}$

2 (b) (i)

Solve the product (**P**) equation for k:

$$1 = \frac{2k+3}{3k}$$

$$3k = 2k + 3$$

$$\therefore k = 3$$

2 (b) (ii)

$$3kx^2 - 18tx + (2k+3) = 0$$

$$\text{Roots: } \alpha, \frac{1}{\alpha} \text{ and } \alpha, 4\alpha$$

$$\mathbf{S}: \alpha + \frac{1}{\alpha} = -\frac{6t}{k} = -2t \ldots.(\mathbf{i})$$

$$\mathbf{S}: \alpha + 4\alpha = -2t \Rightarrow 5\alpha = -2t \Rightarrow \alpha = -\frac{2t}{5}$$

Substitute the value for α into equation (**i**) and solve for t:

$$\left(-\frac{2t}{5}\right) + \left(-\frac{5}{2t}\right) = -2t \ldots.(\mathbf{i})(\times -10t)$$

$$4t^2 + 25 = 20t^2$$

$$25 = 16t^2$$

$$\tfrac{25}{16} = t^2$$

$$\therefore t = \pm\tfrac{5}{4}$$

2 (c) (i)

$f(x)=\dfrac{1}{x^2-6x+13}$ [Prove the expression on the bottom is greater than zero, i.e. positive. One over a positive number is also a positive number.]

$$x^2-6x+13$$
$$=(x^2-6x+9)+4$$
$$=(x-3)^2+4>0 \text{ for all } x\in\mathbb{R}$$
$$\therefore f(x)>0 \text{ for all } x\in\mathbb{R}$$

2 (c) (ii)

$$x^2-6x+13<1$$
$$\Rightarrow (x-3)^2+4<1$$
$$\Rightarrow (x-3)^2<-3$$
$$\therefore x^2-6x+13>1$$
$$\Rightarrow f(x)<1$$

[This is impossible which means that the bottom expression is greater than 1.
The reciprocal of a number greater than 1 is always less than 1.
Therefore, $f(x)<1$.]

2 (c) (iii)

For what values of a is the $f(x)>0$?

$$f(x)>0$$
$$x^2-6x+a>0$$
$$x^2-6x+9+(a-9)>0$$
$$(x-3)^2+(a-9)>0$$
$$(a-9)>0\Rightarrow a>9$$

For what values of a is the $f(x)<1$?

$$f(x)<1$$
$$x^2-6x+a>1$$
$$x^2-6x+9+(a-9)>1$$
$$(x-3)^2+(a-9)>1$$
$$(x-3)^2+(a-10)>0$$
$$(a-10)>0\Rightarrow a>10$$

Answer: $a>10$