ALGEBRA (Q 1 & 2, PAPER 1)

2010

1 (a) $x^2 - 6x + t = (x+k)^2$, where t and k are constants.
Find the value of k and the value of t.

(b) Given that p is a real number, prove that the equation $x^2 - 4px - x + 2p = 0$ has real roots.

(c) $(x - 2)$ and $(x + 1)$ are factors of $x^3 + bx^2 + cx + d$.

(i) Express c in terms of b.

(ii) Express d in terms of b.

(iii) Given that b, c and d are three consecutive terms in an arithmetic sequence, find their values.

SOLUTION

1 (a)

$x^2 - 6x + t = (x+k)^2$

$x^2 - 6x + t = x^2 + 2kx + k^2$

Line up the coefficients:

$2k = -6 \Rightarrow k = -3$

$t = k^2 \Rightarrow t = (3)^2 = 9$

1 (b)

$x^2 - 4px - x + 2p = 0$

$x^2 + (-4p - 1)x + 2p = 0$

$b^2 - 4ac \geq 0 \Rightarrow$ Real roots

$a = 1$
$b = (-4p - 1)$
$c = 2p$

$b^2 - 4ac \geq 0 \Rightarrow (-4p-1)^2 - 4(1)(2p) \geq 0$

$16p^2 + 8p + 1 - 8p \geq 0$

$16p^2 + 1 \geq 0$

This is true for all real values of p. Therefore, this equation has real roots for all real values of p.

1 (c)

$x^3+bx^2+cx+d=(x-2)(x+1)(x+t)$ [A cubic is a linear by a linear by a linear.]

$x^3+bx^2+cx+d=(x^2-x-2)(x+t)$

$x^3+bx^2+cx+d=x^3+tx^2-x^2-2x-tx-2t$

$x^3+bx^2+cx+d=x^3+(t-1)x^2+(-2-t)x-2t$

Line up the coefficients.

$b=t-1$
$\therefore t=b+1$

$c=-2-t$
$c=-2-b-1$
$\therefore c=-b-3...(\text{i})$

$d=-2t$
$\therefore d=-2b-2...(\text{ii})$

1 (c) (iii)

$b,\ c,\ d$ [Three consecutive terms in an arithmetic sequence.]

$b,\ -b-3,\ -2b-2$

$\Rightarrow -b-3-b=-2b-2+b+3$ [Subtracting consecutive terms in an arithmetic sequence gives the common difference d.]

$-2b-3=-b+1$

$-b=4$

$b=-4$

–4, 1, 6 [Replacing b by this value above gives the 3 terms of the sequence.]

2. (a) Solve the simultaneous equations

$$2x+3y=0$$
$$x+y+z=0$$
$$3x+2y-4z=9.$$

(b) The equation $x^2-12x+16=0$ has roots α^2 and β^2, where $\alpha>0$ and $\beta>0$.

(i) Find the value of $\alpha\beta$.

(ii) Hence, find the value of $\alpha+\beta$.

(c) (i) Prove that for all real numbers a and b,

$$a^2-ab+b^2\geq ab.$$

(ii) Let a and b be non-zero real numbers such that $a+b\geq 0$.

Show that $\dfrac{a}{b^2}+\dfrac{b}{a^2}\geq\dfrac{1}{a}+\dfrac{1}{b}.$

SOLUTION

2 (a)

$$2x+3y=0 \text{..........}(\mathbf{1})$$
$$x+y+z=0 \text{......}(\mathbf{2})$$
$$3x+2y-4z=9 \text{...}(\mathbf{3})$$

$$4x+4y+4z=0 \text{...}(\mathbf{2})(\times 4)$$
$$3x+2y-4z=9 \text{...}(\mathbf{3})$$
$$7x+6y \quad =9 \text{...}(\mathbf{4})$$

$$-4x-6y=0 \text{....}(\mathbf{1})(\times -2)$$
$$7x+6y=9 \text{....}(\mathbf{4})$$
$$3x \quad =9 \Rightarrow x=3$$

Substitute into Eqn. (**1**):

$$2x+3y=0 \Rightarrow 2(3)+3y=0$$
$$6+3y=0$$
$$y=-2$$

Substitute into Eqn. (**2**):

$$x+y+z=0 \Rightarrow (3)+(-2)+z=0$$
$$3-2+z=0$$
$$1+z=0$$
$$z=-1$$

Ans: $x = 3$, $y = -2$, $z = -1$

2 (b)

$$x^2-12x+16=0$$
$$\text{Roots}: \alpha^2,\ \beta^2$$
$$\mathbf{S}: \alpha^2+\beta^2=12$$
$$\mathbf{P}: \alpha^2\beta^2=16$$

Sum S: $\alpha+\beta=-\frac{b}{a}=\frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta=\frac{c}{a}=\frac{3^{\text{rd.}}}{1^{\text{st.}}}$

2 (b) (i)

$\alpha^2\beta^2=16 \Rightarrow \alpha\beta=\sqrt{16}=4$ [You are told both roots are positive.]

2 (b) (ii)

$$(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$$
$$\Rightarrow \alpha+\beta=\sqrt{\alpha^2+2\alpha\beta+\beta^2}$$
$$=\sqrt{12+2(4)}=\sqrt{12+8}$$
$$=\sqrt{20}=2\sqrt{5}$$

2 (c) (i)

$a^2 - ab + b^2 \geq ab$

$a^2 - 2ab + b^2 \geq 0$

$(a-b)^2 \geq 0$

2 (c) (ii)

$\frac{a}{b^2} + \frac{b}{a^2} \geq \frac{1}{a} + \frac{1}{b}$ [Multiply across by a^2b^2 - this is legal as you are multiplying across by a positive number.]

$a^3 + b^3 \geq ab^2 + a^2b$

$a^3 + b^3 - ab^2 - a^2b \geq 0$

$a^2(a-b) - b^2(a-b) \geq 0$

$(a-b)(a^2 - b^2) \geq 0$

$(a-b)(a+b)(a-b) \geq 0$

$(a-b)^2(a+b) \geq 0$ [This is true as you are told that $(a + b)$ is greater than or equal to zero.]