ALGEBRA (Q 1 & 2, PAPER 1)

2009

1 (a) Find the value of $\frac{x}{y}$ when $\frac{2x+3y}{x+6y}=\frac{4}{5}$.

(b) Let $f(x)=x^2-7x+12$.

(i) Show that if $f(x+1)\neq 0$, then $\frac{f(x)}{f(x+1)}$ simplifies to $\frac{x-4}{x-2}$.

(ii) Find the range of values of x for which $\frac{f(x)}{f(x+1)}>3$.

(c) Given that $x-c+1$ is a factor of $x^2-5x+5cx-6b^2$, express c in terms of b.

SOLUTION

1 (a)

$$\frac{2x+3y}{x+6y}=\frac{4}{5}$$

$$5(2x+3y)=4(x+6y)$$

$$10x+15y=4x+24y$$

$$6x=9y$$

$$2x=3y\Rightarrow\frac{x}{y}=\frac{3}{2}$$

1 (b) (i)

$$f(x)=x^2-7x+12=(x-4)(x-3)$$

$$f(x+1)=((x+1)-4)((x+1)-3)=(x-3)(x-2)$$

$$\frac{f(x)}{f(x+1)}=\frac{(x-4)(x-3)}{(x-3)(x-2)}=\frac{x-4}{x-2}$$

1 (b) (ii)

$$\frac{f(x)}{f(x+1)}>3\Rightarrow\frac{x-4}{x-2}>3$$

STEPS

1. Multiply both sides by the denominator squared.
2. Get all terms on one side and take out the HCF.
3. Solve the quadratic equation to get the roots.
4. Arrange the roots in ascending order.
5. Carry out the region test to get the solutions.

$$\frac{(x-2)^2(x-4)}{(x-2)} > 3(x-3)^2$$

$$(x-2)(x-4)-3(x-2)^2 > 0$$

$$(x-2)[(x-4)-3(x-2)] > 0$$

$$(x-2)[x-4-3x+6] > 0$$

$$(x-2)(2-2x) > 0$$

$$(x-2)(2-2x) = 0$$

$$\therefore x = 2,\ 1$$

0	1	$\frac{3}{2}$	2	3
$(-2)(2) > 0$		$(-\frac{1}{2})(-1) > 0$		$(1)(-4) > 0$
Wrong		Correct		Wrong

Region Test on $(x-2)(2-2x) > 0$ .....**Test Box**

$\therefore 1 < x < 2$

1 (c)

A quadratic equals a linear by a linear.

$$x^2 + (5c-5)x - 6b^2 = (x+(1-c))(x+k)$$

$$x^2 + (5c-5)x - 6b^2 = x^2 + kx + (1-c)x + k(1-c)$$

$$x^2 + (5c-5)x - 6b^2 = x^2 + (1-c+k)x + k(1-c)$$

$5c-5 = 1-c+k$...**(1)**

$6c-6 = k$

$-6b^2 = k(1-c)$...**(2)**

$-6b^2 = (6c-6)(1-c)$

$-6b^2 = -6(1-c)(1-c)$

$b^2 = (1-c)^2$

$b = \pm(1-c)$

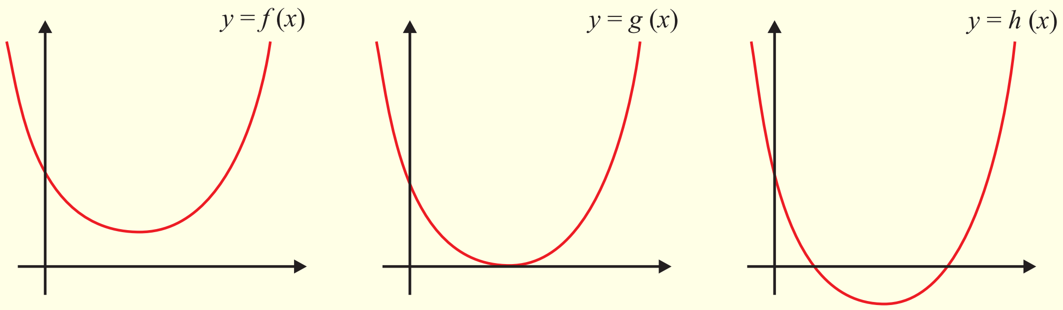
$b = 1-c \Rightarrow c = 1-b$

$b = -1+c \Rightarrow c = 1+b$

2. (a) Solve the simultaneous equations

$$x - y + 8 = 0$$
$$x^2 + xy + 8 = 0.$$

(b) (i) The graphs of three quadratic functions, *f*, *g* and *h*, are shown.



In each case, state the nature of the roots of the function.

(ii) The equation $kx^2 + (1-k)x + k = 0$ has equal roots.
Find the possible values of k.

(c) (i) One of the roots of $px^2 + qx + r = 0$ is n times the other root.
Express r in terms of p, q and n.

(ii) One of the roots of $x^2 + qx + r = 0$ is five times the other.
If q and r are positive integers, determine the set of possible values of q.

Solution

2 (a)

$x - y + 8 = 0....(\mathbf{L})$

$x^2 + xy + 8 = 0....(\mathbf{Q})$

Step 1

$x - y + 8 = 0 \Rightarrow y = (x+8)$

Step 2

$x^2 + xy + 8 = 0$

$x^2 + x(x+8) + 8 = 0$

$x^2 + x^2 + 8x + 8 = 0$

$2x^2 + 8x + 8 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)(x+2) = 0$

$\therefore x = -2$

Step 3

$y = (x+8) = -2 + 8 = 6$

2 (b) (i)

$y = f(x)$: No real roots
$y = g(x)$: Two equal real roots
$y = h(x)$: Two different real roots

2 (b) (ii)

$a = k$
$b = (1-k)$
$c = k$

Condition for equal roots → $b^2 = 4ac$

$$(1-k)^2 = 4(k)(k)$$
$$1-2k+k^2 = 4k^2$$
$$0 = 3k^2+2k-1$$
$$0 = (3k-1)(k+1)$$
$$\therefore k = -1, \tfrac{1}{3}$$

2 (c) (i)

$px^2+qx+r=0$

Roots: α, $n\alpha$

Sum **S**: $\alpha + n\alpha = -\dfrac{q}{p} \Rightarrow (1+n)\alpha = -\dfrac{q}{p} \Rightarrow \alpha = -\dfrac{q}{(1+n)p}$

Product **P**: $n\alpha^2 = \dfrac{r}{p} \Rightarrow n\left[-\dfrac{q}{(1+n)p}\right]^2 = \dfrac{r}{p}$

$$\Rightarrow \frac{nq^2}{(1+n)^2p^2} = \frac{r}{p}$$
$$\Rightarrow \frac{nq^2}{p(1+n)^2} = r$$

$ax^2+bx+c=0$

$\mathbf{S}: \alpha+\beta = -\dfrac{2^{\text{nd}}}{1^{\text{st}}} = -\dfrac{b}{a}$

$\mathbf{P}: \alpha\beta = \dfrac{3^{\text{rd}}}{1^{\text{st}}} = \dfrac{c}{a}$

2 (c) (ii)

$x^2+qx+r=0$

Roots: α, 5α

Sum **S**: $6\alpha = -q \Rightarrow \alpha = -\dfrac{q}{6}$

Product **P**: $5\alpha^2 = r \Rightarrow 5\left(-\dfrac{q}{6}\right)^2 = r$

$$\Rightarrow \frac{5q^2}{36} = r \Rightarrow 5q^2 = 36r$$
$$\Rightarrow q = \sqrt{\frac{36r}{5}} = 6\sqrt{\frac{r}{5}}$$

$q = \{6, 12, 18, \ldots\ldots, 6n\},\ n \in \mathbf{N}_0.$