ALGEBRA (Q 1 & 2, PAPER 1)

2008

1 (a) Simplify fully $\frac{x^2+4}{x^2-4}-\frac{x}{x+2}$.

(b) Given that one of the roots is an integer, solve the equation

$$6x^3-29x^2+36x-9=0.$$

(c) Two of the roots of the equation $ax^3+bx^2+cx+d=0$ are p and $-p$.
Show that $bc = ad$.

SOLUTION

1 (a)

Factorise denominators first.
Get the lowest common denominator (LCD).

Difference of 2 squares

$a^2-b^2=(a+b)(a-b)$

$$\frac{x^2+4}{x^2-4}-\frac{x}{x+2}$$
$$=\frac{x^2+4}{(x+2)(x-2)}-\frac{x}{(x+2)}$$
$$=\frac{1(x^2+4)-x(x-2)}{(x+2)(x-2)}$$
$$=\frac{x^2+4-x^2+2x}{(x+2)(x-2)}$$
$$=\frac{2x+4}{(x+2)(x-2)}$$
$$=\frac{2\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$$
$$=\frac{2}{x-2}$$

1 (b)

STEPS TO SOLVING A CUBIC
1. Find a root by guessing.
2. Get a factor from the root.
3. Find the quadratic by lining up.
4. Factorise or use formula on the resulting quadratic.
5. Write down the three roots.

1. $f(x)=6x^3-29x^2+36x-9$

$f(1)=6(1)^3-29(1)^2+36(1)-9=6-29+36-9=4\neq 0$

$f(3)=6(3)^3-29(3)^2+36(3)-9=162-261+108-9=0$

2. $\therefore (x-3)$ is a factor.

3. Method 1: Division Process

$$\begin{array}{r|l} & 6x^2-11x+3 \\ \hline x-3 & 6x^3-29x^2+36x-9 \\ & \mp 6x^3 \pm 18x^2 \\ \hline & \quad -11x^2+36x-9 \\ & \quad \pm 11x^2 \mp 33x \\ \hline & \qquad 3x-9 \\ & \qquad \mp 3x \pm 9 \\ \hline & \qquad 0 \end{array}$$

$\therefore 6x^3-29x^2+36x-9=(x-3)(6x^2-11x+3)=0$

$\Rightarrow 6x^3-29x^2+36x-9=(x-3)(3x-1)(2x-3)=0$

$\therefore x=3, \frac{1}{3}, \frac{3}{2}$

Method 2: Lining up

> A cubic expression is the product of a quadratic and a linear factor.
> The **first** terms of the linear and quadratics multiply to give the **first** term of the cubic. The **last** terms of the linear and quadratics multiply to give the **last** term of the cubic.

$\therefore 6x^3-29x^2+36x-9=(x-3)(6x^2+kx+3)$

$\Rightarrow 6x^3-29x^2+36x-9=6x^3-18x^2+kx^2+3x-3kx-9$

$\Rightarrow 6x^3-29x^2+36x-9=6x^3+(k-18)x^2+(3-3k)x-9$

$\therefore (k-18)=-29 \Rightarrow k=-11$

$\therefore 6x^3-29x^2+36x-9=(x-3)(6x^2-11x+3)=0$

$\Rightarrow 6x^3-29x^2+36x-9=(x-3)(3x-1)(2x-3)=0$

$\therefore x=3, \frac{1}{3}, \frac{3}{2}$

1 (c)

$x=p \Rightarrow (x-p)$ is a factor.

$x=-p \Rightarrow (x+p)$ is a factor.

$\therefore (x-p)(x+p)=(x^2-p^2)$ is a factor.

Method 1: Division process

$$\begin{array}{r|l} & ax+b \\ \hline x^2-p^2 & ax^3+bx^2+cx+d \\ & \mp ax^3 \qquad \pm ap^2x \\ \hline & \qquad bx^2+(c+ap^2)x+d \\ & \qquad \mp bx^2 \qquad\qquad \pm p^2b \\ \hline & \qquad\qquad (c+ap^2)x+(d+p^2b) \end{array}$$

The remainder must be zero, i.e. $0x + 0$

$\therefore (c + ap^2) = 0$ and $(d + p^2b) = 0$

$c = -ap^2 \Rightarrow p^2 = -\frac{c}{a}$

$d = -p^2b \Rightarrow d = (\frac{c}{a})b$

$\therefore ad = bc$

METHOD 2: Lining up

$ax^3 + bx^2 + cx + d = (x^2 - p^2)(ax + k)$

$\Rightarrow ax^3 + bx^2 + cx + d = ax^3 + kx^2 - ap^2x - p^2k$

$\therefore k = b$

$\therefore c = -ap^2 \Rightarrow p^2 = -\frac{c}{a}$

$\therefore d = -p^2k \Rightarrow d = (\frac{c}{a})b \Rightarrow ad = bc$

2. (a) Express $x^2 + 10x + 32$ in the form $(x + a)^2 + b$.

(b) α and β are the roots of the equation $x^2 - 7x + 1 = 0$.

(i) Find the value of $\alpha^2 + \beta^2$.

(ii) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$.

(c) Show that if a and b are non-zero real numbers, then the value of $\frac{a}{b} + \frac{b}{a}$ can never lie between -2 and 2.

HINT: Consider the case where a and b have the same sign separately from the case where a and b have opposite signs.

SOLUTION

2 (a)

To complete the square, take half of the coefficient of x and square it.

$x^2 + 10x + 32$

$= x^2 + 10x + (5)^2 + 7$

$= (x + 5)^2 + 7$

2 (b) (i)

$x^2 - 7x + 1 = 0$

Sum **S**: $\alpha + \beta = 7$

Product **P**: $\alpha\beta = 1$

$\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (7)^2 - 2(1) = 47$

Sum S: $\alpha + \beta = -\frac{b}{a} = \frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta = \frac{c}{a} = \frac{3^{\text{rd.}}}{1^{\text{st.}}}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

2 (b) (ii)

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$$

$$= \frac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{\alpha^3\beta^3}$$

$$= \frac{(7)(47 - 1)}{(1)^3} = (7)(46) = 322$$

2 (c)

Method 1: Let $\frac{a}{b} + \frac{b}{a} = x \Rightarrow a^2 + b^2 = abx$

$\therefore a^2 - (bx)a + b^2 = 0$

This equation is a quadratic in a.

Remember: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

It has real solutions if $b^2 - 4ac \geq 0$.

Note: Don't mix up the letters. They have different meanings in different situations.

$a = 1$
$b = -bx$
$c = b^2$

$b^2 - 4ac \geq 0 \Rightarrow b^2x^2 - 4(1)(b^2) \geq 0$

$\Rightarrow b^2x^2 - 4b^2 \geq 0$

$\Rightarrow x^2 - 4 \geq 0$

$\Rightarrow (x + 2)(x - 2) \geq 0$

Solve $(x + 2)(x - 2) = 0 \Rightarrow x = -2,\ 2$

Carry out the region test:

←	-2	↔	2	→
-3		0		3
$(-1)(-5) \geq 0$		$(2)(-2) \geq 0$		$(5)(1) \geq 0$
Correct		Wrong		Correct

Test box: $(x + 2)(x - 2) \geq 0$

$\therefore x < -2,\ x > 2$

METHOD 2: Using the hint given

Same signs: If a and b have the same sign, either positive or negative, this means that each fraction is positive and you will have to show their sum has to be greater that +2.

To prove:

$\frac{a}{b}+\frac{b}{a}\geq 2$ [Multiply across by ab. This is legal as ab is positive.]

$\Rightarrow a^2+b^2\geq 2ab$

$\Rightarrow a^2-2ab+b^2\geq 0$

$\Rightarrow (a-b)^2\geq 0$ [This is true.]

Different signs: If a and b have different signs, this means that each fraction is negative and you will have to show that their sum is less than –2.

$\frac{a}{b}+\frac{b}{a}\leq -2$ [Multiply across by ab. As ab is negative, you have to reverse the inequality.]

$\Rightarrow a^2+b^2\geq -2ab$

$\Rightarrow a^2+2ab+b^2\geq 0$

$\Rightarrow (a+b)^2\geq 0$ [This is true.]

Therefore, the value does not lie between 2 and –2.