ALGEBRA (Q 1 & 2, PAPER 1)

2000

1 (a) Show that the following simplifies to a constant when $x \neq 2$

$$\frac{3x-5}{x-2}+\frac{1}{2-x}.$$

(b) $f(x)=ax^3+bx^2+cx+d$ where $a,b,c,d \in \mathbf{R}$.

If k is a real number such that $f(k)=0$, prove that $x-k$ is a factor of $f(x)$.

(c) $(x-t)^2$ is a factor of $x^3+3px+c$.

Show that

(i) $p=-t^2$

(ii) $c=2t^3$.

SOLUTION

1 (a)

$$\frac{3x-5}{x-2}+\frac{1}{2-x}$$

$$=\frac{3x-5}{(x-2)}-\frac{1}{(x-2)}$$

$$=\frac{3x-5-1}{(x-2)}=\frac{3x-6}{(x-2)}$$

$$=\frac{3\cancel{(x-2)}}{\cancel{(x-2)}}=3$$

Factorise denominators first.
Get the lowest common denominator (LCD).

1 (b)

PROOF OF FACTOR THEOREM FOR A CUBIC

$f(x)=ax^3+bx^2+cx+d$

$f(k)=ak^3+bk^2+ck+d$

$\therefore f(x)-f(k)=a(x^3-k^3)+b(x^2-k^2)+c(x-k)$

$=(x-k)\{ax^2+akx+ak^2+bx+bk+c\}=(x-k)g(x)$

$\therefore f(x)=f(k)+(x-k)g(x)$

(i) $f(k)=0 \Rightarrow f(x)=(x-k)g(x) \therefore x-k$ is a factor.

(ii) $x-k$ is a factor $\Rightarrow f(k)=0$.

Method 1: Division Process

1 (c)

$$
\begin{array}{r|l}
 & x+2t \\
\hline
x^2-2tx+t^2 & x^3+0x^2+3px+c \\
 & \mp x^3 \pm 2tx^2 \mp t^2x \\
\cline{2-2}
 & \qquad 2tx^2+(3p-t^2)x+c \\
 & \qquad \mp 2tx^2 \pm 4t^2x \quad \mp 2t^3 \\
\cline{2-2}
 & \qquad\qquad (3p+3t^2)x+(c-2t^3)
\end{array}
$$

The remainder has to be zero, i.e. $0x + 0$.

1 (c) (i)

$\therefore (3p+3t^2)=0 \Rightarrow p=-t^2$

1 (c) (ii)

$\therefore (c-2t^3)=0 \Rightarrow c=2t^3$

Method 2: Lining up

1 (c)

A cubic is a quadratic multiplied by a linear. The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear. Also, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear.

$\therefore x^3+0x^2+3px+c=(x^2-2tx+t^2)(x+k)$

$\Rightarrow x^3+0x^2+3px+c=x^3+kx^2-2tx^2-2tkx+t^2x+t^2k$

$\Rightarrow x^3+0x^2+3px+c=x^3+(k-2t)x^2-(2tk+t^2)x+t^2k$

Lining up the coefficients:

$\therefore (k-2t)=0$ and $(-2tk+t^2)=3p$ and $t^2k=c$

Eliminate the k as this was introduced by us and substitute its value into the other identities. Therefore, $k = 2t$.

1 (c) (i)

$-2tk+t^2=3p \Rightarrow -2t(2t)+t^2=3p \Rightarrow -4t^2+t^2=3p$

$\Rightarrow -3t^2=3p$

$\therefore p=-t^2$

1 (c) (ii)

$t^2k=c \Rightarrow t^2(2t)=c$

$\therefore c=2t^3$

2 (a) Solve for x, y, z

$$3x - y + 3z = 1$$
$$x + 2y - 2z = -1$$
$$4x - y + 5z = 4$$

(b) Solve $x^2 - 2x - 24 = 0$.
Hence, find the values of x for which

$$\left(x + \frac{4}{x}\right)^2 - 2\left(x + \frac{4}{x}\right) - 24 = 0,\ x \in \mathbf{R},\ x \neq 0.$$

(c) (i) Express $a^4 - b^4$ as a product of three factors.

(ii) Factorise $a^5 - a^4 b - ab^4 + b^5$.
Use your results from **(i)** and **(ii)** to show that
$a^5 + b^5 > a^4 b + ab^4$
where a and b are positive unequal real numbers.

SOLUTION

2 (a)

Eliminate y from equations **1** and **3**:

$$3x - y + 3z = 1.....(\mathbf{1})$$
$$x + 2y - 2z = -1..(\mathbf{2})$$
$$4x - y + 5z = 4.....(\mathbf{3})$$

$$3x - y + 3z = 1.....(\mathbf{1})(\times -1)$$
$$4x - y + 5z = 4.....(\mathbf{3})$$

$$-3x + y - 3z = -1$$
$$4x - y + 5z = 4$$
$$x \quad + 2z = 3.....(\mathbf{4})$$

Eliminate y from equations **1** and **2**:

$$3x - y + 3z = 1.....(\mathbf{1})(\times 2)$$
$$x + 2y - 2z = -1..(\mathbf{2})$$

$$6x - 2y + 6z = 2$$
$$x + 2y - 2z = -1$$
$$7x \quad + 4z = 1......(\mathbf{5})$$

Now combine equations **4** and **5** to eliminate z:

$$x + 2z = 3...(\mathbf{4})(\times -7)$$
$$7x + 4z = 1...(\mathbf{5})$$

$$-7x - 14z = -21$$
$$7x + 4z = 1$$
$$-10z = -20 \Rightarrow z = 2$$

Substituting this value of z into equation **4**

$\therefore x + 2(2) = 3 \Rightarrow x + 4 = 3 \Rightarrow x = -1$

Substituting these values of x and z into equation **1**

$\Rightarrow 3(-1) - y + 3(2) = 1 \Rightarrow -3 - y + 6 = 1 \Rightarrow y = 2$

ANSWER: $x = -1,\ y = 2,\ z = 2$

2 (b)

$x^2-2x-24=0$

$\Rightarrow (x+4)(x-6)=0$

$\therefore x=-4,\ x=6$

$\left(x+\frac{4}{x}\right)^2-2\left(x+\frac{4}{x}\right)-24=0$

$\therefore \left(x+\frac{4}{x}\right)=-4 \Rightarrow x^2+4=-4x$

$\Rightarrow x^2+4x+4=0$

$\Rightarrow (x+2)(x+2)=0$

$\therefore x=-2$

$\therefore \left(x+\frac{4}{x}\right)=6 \Rightarrow x^2+4=6x$

$\Rightarrow x^2-6x+4=0$

$\Rightarrow x=\frac{6\pm\sqrt{(-6)^2-4(1)(4)}}{2}=\frac{6\pm\sqrt{20}}{2}=\frac{6\pm2\sqrt{5}}{2}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=3\pm\sqrt{5}$

Answer: $x=-2,\ 3\pm\sqrt{5}$

2 (c) (i)

$a^4-b^4=(a^2)^2-(b^2)^2$

$a^2-b^2=(a+b)(a-b)$

$=(a^2+b^2)(a^2-b^2)$

$=(a^2+b^2)(a+b)(a-b)$

2 (c) (ii)

$a^5-a^4b-ab^4+b^5$

$=a^4(a-b)-b^4(a-b)$

$=(a^4-b^4)(a-b)$

$=(a^2+b^2)(a+b)(a-b)(a-b)$

$=(a^2+b^2)(a+b)(a-b)^2$

$a^5+b^5>a^4b+ab^4$

$\Rightarrow a^5-a^4b-ab^4+b^5>0$

$\Rightarrow (a^2+b^2)(a+b)(a-b)^2>0$

This statement is true for all *a* and *b* that are positive and unequal.