ALGEBRA (Q 1 & 2, PAPER 1)

1999

1 (a) Show that $\dfrac{-1+\sqrt{3}}{1+\sqrt{3}} = 2-\sqrt{3}$.

(b) Solve for x

$$\frac{4x-1}{x-3} < 2,\ x \in \mathbf{R} \text{ and } x \neq 3.$$

(c) $x^2 + bx + c$ is a factor of $x^3 - p$.
Show that

(i) $b^3 = p$

(ii) $c^3 = p^2$.

SOLUTION

1 (a)

$$\frac{-1+\sqrt{3}}{1+\sqrt{3}} = \frac{(-1+\sqrt{3})}{(1+\sqrt{3})} \times \frac{(1-\sqrt{3})}{(1-\sqrt{3})}$$

$$= \frac{-1+\sqrt{3}+\sqrt{3}-3}{1-3}$$

$$= \frac{-4+2\sqrt{3}}{-2} = 2-\sqrt{3}$$

SURDS
You are often asked to rationalise the denominator of a surd. This means getting rid of surds in the denominator. Multiply above and below by the conjugate of the denominator.

1 (b)

STEPS
1. Move all terms to the same side so that you have a positive coefficient of x^2.
2. Try to factorise the quadratic.
3. Solve the quadratic equation to get the roots.
4. Arrange the roots in ascending order.
5. Carry out the region test to get the solutions.

$\dfrac{4x-1}{x-3} < 2$ [Multiply both sides by the denominator squared.]

$\Rightarrow (4x-1)(x-3) < 2(x-3)^2$

$\Rightarrow (4x-1)(x-3) - 2(x-3)^2 < 0$ [Factorise the left-hand side]

$\Rightarrow (x-3)[(4x-1) - 2(x-3)] < 0$

$\Rightarrow (x-3)[4x-1-2x+6] < 0$

$\Rightarrow (x-3)(2x+5) < 0$

Solve the equality: $(x-3)(2x+5) = 0 \Rightarrow x = -\frac{5}{2}, 3$

Do the region test:

-3	$-\frac{5}{2}$	0	3	4
$(-3-3)(-6+5)<0$		$(0-3)(0+5)<0$		$(4-3)(4+5)<0$
Wrong		Correct		Wrong

Region Test on $(x-3)(2x+5)<0$ .....**Test Box**

$\therefore -\frac{5}{2}<x<3$

1 (c)

METHOD 1: DIVISION PROCESS

$$
\begin{array}{r|l}
 & x-b \\
x^2+bx+c & x^3+0x^2+0x-p \\
 & \mp x^3 \mp bx^2 \mp cx \\
 & -bx^2-cx-p \\
 & \pm bx^2 \pm b^2x \pm bc \\
 & (b^2-c)x+(bc-p)
\end{array}
$$

The remainder has to be zero, i.e. 0x + 0.

$\therefore (b^2-c)=0 \Rightarrow b^2=c$

$\therefore (bc-p)=0 \Rightarrow bc=p$

1 (c) (i)

$bc=p \Rightarrow b(b^2)=p$

$\therefore b^3=p$

1 (c) (ii)

$b^2=c \Rightarrow b^6=c^3 \Rightarrow (b^3)^2=c^3$

$\therefore c^3=p^2$

METHOD 2: LINING UP

1 (c)

A cubic is a quadratic multiplied by a linear. The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear. Also, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear.

$x^3+0x^2+0x-p=(x^2+bx+c)(x+k)$

$\Rightarrow x^3+0x^2+0x-p=x^3+(b+k)x^2+(bk+c)x+k$

Lining up the coefficients:

$\therefore b+k=0$ and $bk+c=0$ and $ck=-p$.

Eliminate the k as this was introduced by us and substitute its value into the other identities.

Therefore, $k=-b$.

1 (c) (i)

$bk + c = 0 \Rightarrow -b^2 + c = 0 \Rightarrow c = b^2$

$ck = -p \Rightarrow -bc = -p \Rightarrow b^3 = p$

1 (c) (ii)

$b^2 = c \Rightarrow b^6 = c^3 \Rightarrow (b^3)^2 = c^3$

$\therefore c^3 = p^2$

2 (a) Solve the simultaneous equations

$$x + y = 1$$
$$x^2 + y^2 = 25$$

(b) If for all integers n,

$$u_n = 2^{2n-1} + 2^{n-1},$$

show that $u_{n+1} - 2u_n - 2^{2n} = 0.$

(c) Let a, b, c be positive, unequal real numbers.
Using the results $a^2 + b^2 > 2ab,\ b^2 + c^2 > 2bc,\ c^2 + a^2 > 2ac,$

(i) deduce that $a^2 - ab + b^2 > ab$

(ii) deduce that $a^2 + b^2 + c^2 > bc + ca + ab$

(iii) show that $a^3 + b^3 > ab(a + b).$

SOLUTION

2 (a)

SOLVING SIMULTANEOUS LINEAR EQUATIONS
1. Get a letter on its own from the linear equation.
2. Substitute into the quadratic and solve.
3. Substitute these values back into linear.

From the linear equation $x = 1 - y$

Substitute for x in the quadratic equation:

$(1-y)^2 + y^2 = 25 \Rightarrow 1 - 2y + y^2 + y^2 = 25$

$\Rightarrow 2y^2 - 2y - 24 = 0 \Rightarrow y^2 - y - 12 = 0$

$\Rightarrow (y-4)(y+3) = 0$

$\therefore y = -3,\ 4$

$\Rightarrow x = 4,\ -3$

ANS: $(-3,\ 4),\ (4,\ -3)$

2 (b)

$u_n = 2^{2n-1} + 2^{n-1}$

$\therefore u_{n+1} = 2^{2(n+1)-1} + 2^{(n+1)-1} = 2^{2n+1} + 2^n$

$u_{n+1} - 2u_n - 2^{2n} = 2^{2n+1} + 2^n - 2(2^{2n-1} + 2^{n-1}) - 2^{2n}$

$= 2^{2n+1} + \cancel{2^n} - 2^{2n} - \cancel{2^n} - 2^{2n}$

$= 2^{2n+1} - 2(2^{2n})$

$= 2^{2n+1} - 2^{2n+1}$

$= 0$

2 (c) (i)

$a^2 + b^2 > 2ab$

$\Rightarrow a^2 + b^2 > ab + ab$

$\Rightarrow a^2 - ab + b^2 > ab$

2 (c) (ii)

Add the three inequalities together:

$\therefore a^2 + b^2 + b^2 + c^2 + c^2 + a^2 > 2ab + 2bc + 2ac$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 > 2ab + 2bc + 2ac$

$\Rightarrow a^2 + b^2 + c^2 > bc + ca + ab$

2 (c) (iii)

Sum of 2 cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 + b^3 > ab(a+b)$ [Factorise the left hand side]

$\Rightarrow (a+b)(a^2 - ab + b^2) > ab(a+b)$ [Divide across by (a + b).]

$\Rightarrow (a^2 - ab + b^2) > ab$ [This was proved in **2 (c) (i)**.]

Note: Dividing across by (a + b) is legal as a and b are positive numbers.