ALGEBRA (Q 1 & 2, PAPER 1)

1998

1 (a) Solve for x and y:

$$\frac{2x-5}{3}+\frac{y}{5}=6$$

$$\frac{3x}{10}+2=\frac{3y-5}{2}.$$

(b) If $(2x-1)$ is a factor of the polynomial

$$P(x)=2x^3-5x^2-kx+3,$$

find the value of k.
Find the other two factors of $P(x)$.

(c) If the quadratic equation $ax^2+bx+c=0$ has equal roots, solve for x in terms of a and b, where $a,b,c\in\mathbf{R}$.

By letting $x=3^y$, write

$$t3^y+3^{-y}=3$$

as a quadratic equation in x, where $t\in\mathbf{R}$ and $t\neq 0$.
Find the value of t for which this equation has equal roots.
Assuming this value of t, solve the equation

$$t3^y+3^{-y}=3.$$

SOLUTION

1 (a)

SOLVING SIMULTANEOUS LINEAR EQUATIONS

1. Get the x's on y's on one side and the number on the other side.
2. Eliminate either the x's or y's.
3. Solve for the remaining letter.
4. Substitute to get the other letter.

$\frac{2x-5}{3}+\frac{y}{5}=6$ [Multiply each term by 15]

$\Rightarrow 5(2x-5)+3y=90$

$\Rightarrow 10x-25+3y=90$

$\therefore 10x+3y=115......(\mathbf{1})$

$\frac{3x}{10}+2=\frac{3y-5}{2}$ [Multiply each term by 10]

$\Rightarrow 2x+20=5(3y-5)$

$\Rightarrow 2x+20=15y-25$

$\Rightarrow 2x-15y=-45....(\mathbf{2})$

$10x+3y=115.....(\mathbf{1})(\times 5)$

$2x-15y=-45....(\mathbf{2})$

$\rightarrow$

$50x+15y=565$

$2x-15y=-45$

$52x=520\Rightarrow x=10$

Substitute this value of x back into Eqn. (**1**).

$\therefore 2(10)-15y=-45\Rightarrow -15y=-65\Rightarrow y=5$

Ans: $x=10$, $y=5$

1 (b)

The factor theorem states that:

If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$, i.e. $f(k) = 0$ and vice versa.

If $(2x-1)$ is a factor of $P(x)$, then $\frac{1}{2}$ is a root of $P(x) = 0$, i.e. $P(\frac{1}{2}) = 0$.

$\therefore 2(\frac{1}{2})^3 - 5(\frac{1}{2})^2 - k(\frac{1}{2}) + 3 = 0$

$\Rightarrow \frac{1}{4} - \frac{5}{4} - \frac{1}{2}k + 3 = 0$

$\Rightarrow 1 - 5 - 2k + 12 = 0$

$\Rightarrow -2k = -8$

$\therefore k = 4$

You can find the two other factors using the division process or by lining up coefficients.

Division Process:

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ 2x-1 & 2x^3 - 5x^2 - 4x + 3 \\ & \mp 2x^3 \pm x^2 \\ & \quad -4x^2 - 4x + 3 \\ & \quad \pm 4x^2 \mp 2x \\ & \qquad -6x + 3 \\ & \qquad \pm 6x \mp 3 \\ & \qquad\quad 0 \end{array}$$

$\therefore 2x^3 - 5x^2 - 4x + 3 = (2x-1)(x^2 - 2x - 3)$

$\Rightarrow 2x^3 - 5x^2 - 4x + 3 = (2x-1)(x-3)(x+1)$

Lining up:

A cubic expression is the product of a quadratic and a linear factor.
The **first** terms of the linear and quadratics multiply to give the **first** term of the cubic. The **last** terms of the linear and quadratics multiply to give the **last** term of the cubic.

$\therefore 2x^3 - 5x^2 - 4x + 3 = (2x-1)(x^2 + ax - 3)$

$\Rightarrow 2x^3 - 5x^2 - 4x + 3 = 2x^3 + (2a-1)x^2 + (-a-6)x + 3$

You can solve for a by lining up the coefficients: $2a - 1 = -5 \Rightarrow 2a = -4 \Rightarrow a = -2$

$\therefore 2x^3 - 5x^2 - 4x + 3 = (2x-1)(x^2 - 2x - 3)$

$\Rightarrow 2x^3 - 5x^2 - 4x + 3 = (2x-1)(x-3)(x+1)$

1 (c)

Quadratic: $ax^2 + bx + c = 0$
Equal roots: α, α

Sum S: $2\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{2a}$

Product P: $\alpha^2 = \frac{c}{a} \Rightarrow (-\frac{b}{2a})^2 = \frac{c}{a}$

$$\Rightarrow \frac{b^2}{4a^2} = \frac{c}{a} \Rightarrow b^2 = 4ac$$

Sum S: $\alpha + \beta = -\frac{b}{a} = \frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta = \frac{c}{a} = \frac{3^{\text{rd.}}}{1^{\text{st.}}}$

Equal roots means that the discriminant, $b^2 - 4ac$, is zero.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = -\frac{b}{2a}$$

$$t3^y + 3^{-y} = 3$$

$$\Rightarrow t3^y - 3 + \frac{1}{3^y} = 0 \quad \text{[Let } x = 3^y \text{]}$$

$$\Rightarrow tx - 3 + \frac{1}{x} = 0 \quad \text{[Multiply each term by } x\text{.]}$$

$$\Rightarrow tx^2 - 3x + 1 = 0$$

The condition for equal roots is that $b^2 = 4ac$.

$a = t$
$b = -3$
$c = 1$

$$\therefore (-3)^2 = 4(t)(1) \Rightarrow 9 = 4t$$

$$\therefore t = \tfrac{9}{4}$$

The solution to a quadratic equation with equal roots is:

$$x = -\frac{b}{2a} = -\frac{(-3)}{2t} = \frac{3}{2t}$$

$$t = \tfrac{9}{4} \Rightarrow x = \frac{3}{2(\frac{9}{4})} = \frac{2}{3}$$

$$x = 3^y \Rightarrow \tfrac{2}{3} = 3^y \quad \text{[Take the log to base 3 of both sides.]}$$

$$\Rightarrow \log_3(\tfrac{2}{3}) = \log_3 3^y$$

$$\Rightarrow \log_3(\tfrac{2}{3}) = y \log_3 3$$

$$\Rightarrow y = \log_3(\tfrac{2}{3})$$

Properties of logs: These enable you to manipulate logs.

Log Rules

3. $N \log_a M = \log_a (M^N)$
5. $\log_a a = 1$

2 (a) Solve for x: $|x-4|<5$.

(b) If α and β are the roots of the equation

$x^2-6x+2=0,\ \alpha>0,\ \beta>0,$

find $\alpha\beta$ and $\alpha+\beta$.

Factorise $\alpha^3+\beta^3$.

Find the value of $\alpha^3+\beta^3$.

(c) Show for all real numbers a, $b>0$, that

$a+b\geq 2\sqrt{ab}$.

Show that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{(a+b)^2}{ab}$.

Deduce that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$.

SOLUTION

2 (a)

STEPS
1. Solve the modulus equation to get the roots.
2. Arrange the roots in ascending order.
3. Carry out the region test to get the solutions.

1. Solve $|x-4|=5$.

$\therefore x-4=\pm 5\Rightarrow x=-1,\ 9$

2. Region test:

Region Test on $|x-4|<5$ .....**Test Box**

$x<-1$	$-1<x<9$	$x>9$
-2	0	10
$\lvert -6\rvert<5$	$\lvert -4\rvert<5$	$\lvert 6\rvert<5$
Wrong	Correct	Wrong

3. $\therefore -1<x<9$

2 (b)

Quadratic: $x^2-6x+2=0$

Roots: $\alpha,\ \beta$

Sum S: $\alpha+\beta=-(\frac{-6}{1})=6$

Product P: $\alpha\beta=(\frac{2}{1})=2$

$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

Sum S: $\alpha+\beta=-\frac{b}{a}=\frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta=\frac{c}{a}=\frac{3^{\text{rd.}}}{1^{\text{st.}}}$

Sum of 2 cubes

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$$
$$=(\alpha+\beta)((\alpha+\beta)^2-2\alpha\beta-\alpha\beta)$$
$$=(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)$$
$$=(6)((6)^2-3(2))$$
$$=6(36-6)=6(30)$$
$$=180$$

These results are useful answering questions on quadratics:

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$$
$$\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$$

2 (c)

$a+b\geq 2\sqrt{ab}$ [Square both sides.]

$\Rightarrow (a+b)^2\geq 4ab$

$\Rightarrow a^2+2ab+b^2-4ab\geq 0$

$\Rightarrow a^2-2ab+b^2\geq 0$

$\Rightarrow (a-b)^2\geq 0$ [This is always true.]

Note: It is legal to square both sides as you are told that both *a* and *b* are positive.

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=(a+b)\left(\frac{b+a}{ab}\right)=\frac{(a+b)^2}{ab}$$

From the previous inequality it is shown that $(a+b)^2\geq 4ab$.

$$\therefore (a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{(a+b)^2}{ab}\geq\frac{4\cancel{ab}}{\cancel{ab}}$$

$$\Rightarrow (a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$$