ALGEBRA (Q 1 & 2, PAPER 1)

1997

1 (a) If $x=\sqrt{a}+\frac{1}{\sqrt{a}}$ and $y=\sqrt{a}-\frac{1}{\sqrt{a}}$, $a>0$, find the value of $\sqrt{x^2-y^2}$.

(b) Let $f(x)=ax^3+bx^2+cx+d$, where $a,b,c,d\in\mathbf{R}$. If k is a real number such that $f(k)=0$, prove that $x-k$ is a factor of $f(x)$.

(c) If $(x-1)^2$ is a factor of ax^3+bx^2+1, find the value of a and the value of b.

SOLUTION

1 (a)

Difference of 2 squares

$a^2-b^2=(a+b)(a-b)$

$\sqrt{x^2-y^2}=\sqrt{(x+y)(x-y)}$

$$=\sqrt{\left(\sqrt{a}+\frac{1}{\sqrt{a}}+\sqrt{a}-\frac{1}{\sqrt{a}}\right)\left(\sqrt{a}+\frac{1}{\sqrt{a}}-\sqrt{a}+\frac{1}{\sqrt{a}}\right)}$$

$$=\sqrt{(2\sqrt{a})\left(\frac{2}{\sqrt{a}}\right)}=\sqrt{4}=2$$

1 (b)

PROOF OF FACTOR THEOREM FOR A CUBIC

$f(x)=ax^3+bx^2+cx+d$

$f(k)=ak^3+bk^2+ck+d$

$\therefore f(x)-f(k)=a(x^3-k^3)+b(x^2-k^2)+c(x-k)$

$=(x-k)\{ax^2+akx+ak^2+bx+bk+c\}=(x-k)g(x)$

$\therefore f(x)=f(k)+(x-k)g(x)$

(i) $f(k)=0\Rightarrow f(x)=(x-k)g(x)\ \therefore x-k$ is a factor.

(ii) $x-k$ is a factor $\Rightarrow f(k)=0$.

1 (c)

METHOD 1: Division process

$$\begin{array}{r|l} & ax+(2a+b) \\ x^2-2x+1 & ax^3+bx^2+0x+1 \\ & \mp ax^3\pm 2ax^2\mp ax \\ & \qquad (2a+b)x^2-ax+1 \\ & \qquad \mp(2a+b)x^2\pm 2(2a+b)x\mp(2a+b) \\ & \qquad\qquad (3a+2b)x+(1-2a-b) \end{array}$$

The remainder must be zero, i.e. $0x+0$.

$\therefore 3a+2b=0$ and $1-2a-b=0$.

Solve for a and b simultaneously.

$3a+2b=0 \ldots(\mathbf{1})$
$2a+b \;=1 \ldots(\mathbf{2})(\times -2)$

$\rightarrow$

$3a+2b=0$
$-4a-2b=-2$
$-a \quad\quad =-2 \Rightarrow a=2$

Substitute this value of a into Eqn. (**2**): $2(2)+b=1 \Rightarrow b=-3$

Ans: $a=2,\ b=-3$

Method 2: Lining up (much better)
A cubic expression is the product of a quadratic and a linear factor.
The **first** terms of the linear and quadratics multiply to give the **first** term of the cubic. The **last** terms of the linear and quadratics multiply to give the **last** term of the cubic.

$$ax^3+bx^2+0x+1=(x^2-2x+1)(ax+1)$$
$$\Rightarrow ax^3+bx^2+0x+1=ax^3+x^2-2ax^2-2x+ax+1$$
$$\Rightarrow ax^3+bx^2+0x+1=ax^3+(1-2a)x^2+(a-2)x+1$$

Lining up: $a-2=0 \Rightarrow a=2$

$1-2a=b \Rightarrow 1-4=b \Rightarrow b=-3$

Why would anyone do this question by the division process?

2 (a) Solve the simultaneous equations

$$2x-3y=1$$
$$x^2+xy-4y^2=2$$

(b) Solve

$$x^2-6x+8=0$$

and hence find the values of x for which

$$(x+\tfrac{1}{x})^2-6(x+\tfrac{1}{x})+8=0, x\in\mathbf{R} \text{ and } x\neq 0.$$

(c) Let $f(x)=\frac{1}{x}$ for all $x\in\mathbf{R}$ and $x\neq 0$.
Points a and b have coordinates $(p, f(p))$ and $(q, f(q))$, respectively, for $0<p<q$.

(i) Show that the equation of the line ab can be written as

$$y=g(x)=\tfrac{1}{p}-\tfrac{1}{pq}(x-p).$$

(ii) Show that

$$f(x)-g(x)=\frac{(x-q)(x-p)}{pqx}.$$

Hence, show that $f(x)-g(x)<0$ for $0<p<x<q$.

SOLUTION

2 (a)

SIMULTANEOUS LINEAR AND QUADRATIC EQUATIONS
1. Get a letter on its own from the linear equation.
2. Substitute into the quadratic and solve.
3. Substitute these values back into linear.

$$2x-3y=1 \Rightarrow 2x=3y+1$$
$$\Rightarrow x=\frac{3y+1}{2}$$

$\rightarrow$

$$x^2+xy-4y^2=2$$
$$\Rightarrow \left(\frac{3y+1}{2}\right)^2+\left(\frac{3y+1}{2}\right)y-4y^2=2$$
$$\Rightarrow \frac{9y^2+6y+1}{4}+\frac{3y^2+y}{2}-4y^2=2$$
$$\Rightarrow 9y^2+6y+1+6y^2+2y-16y^2=8$$
$$\Rightarrow -y^2+8y-7=0$$
$$\Rightarrow y^2-8y+7=0$$
$$\Rightarrow (y-7)(y-1)=0$$
$$\therefore y=1,\ 7$$

$\leftarrow$

$$y=1: x=\frac{3(1)+1}{2}=2$$
$$y=7: x=\frac{3(7)+1}{2}=11$$

Ans: (2, 1), (11, 7)

2 (b)

$$x^2-6x+8=0$$
$$\Rightarrow (x-4)(x-2)=0$$
$$\therefore x=2,\ 4$$
$$\left(x+\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)+8=0$$
$$\Rightarrow \left(x+\frac{1}{x}\right)=2 \text{ and } \left(x+\frac{1}{x}\right)=4$$

Solve $x+\frac{1}{x}=2$
$$\Rightarrow x^2+1=2x$$
$$\Rightarrow x^2-2x+1=0$$
$$\Rightarrow (x-1)(x-1)=0$$
$$\therefore x=1$$

Solve $x+\frac{1}{x}=4$
$$\Rightarrow x^2+1=4x$$
$$\Rightarrow x^2-4x+1=0$$
$$\Rightarrow x=\frac{4\pm\sqrt{16-4(1)(1)}}{2}=\frac{4\pm\sqrt{12}}{2}$$
$$\Rightarrow x=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Ans: $x=1,\ 2\pm\sqrt{3}$

2 (c)

$f(x) = \frac{1}{x}$

$\therefore a(p,\ f(p)) = a(p,\ \frac{1}{p})$

$\therefore b(q,\ f(q)) = b(q,\ \frac{1}{q})$

2 (c) (i)

Equation of *ab*:

$m = \dfrac{\frac{1}{q} - \frac{1}{p}}{q-p}$ [Multiply above and below by *pq*.]

$m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow m = \dfrac{(\frac{1}{q} - \frac{1}{p})}{q-p} \times \dfrac{pq}{pq} = \dfrac{p-q}{pq(q-p)} = \dfrac{-(q-p)}{pq\,(q-p)} = -\dfrac{1}{pq}$

Slope $m = -\dfrac{1}{pq}$, point $(x_1,\ y_1) = (p,\ \frac{1}{p})$

$y - y_1 = m(x - x_1)$

$y - \dfrac{1}{p} = -\dfrac{1}{pq}(x-p)$

$\Rightarrow y = \dfrac{1}{p} - \dfrac{1}{pq}(x-p)$ [*y* is also called *g*(*x*).]

2 (c) (ii)

$f(x) - g(x) = \dfrac{1}{x} - \dfrac{1}{p} + \dfrac{1}{pq}(x-p)$

$= \dfrac{pq - qx + x(x-p)}{pqx}$

$= \dfrac{pq - qx + x^2 - px}{pqx} = \dfrac{x^2 - (p+q)x + pq}{pqx}$

$= \dfrac{(x-q)(x-p)}{pqx}$

$(x-q) < 0$ as $x < q$

$(x-p) > 0$ as $p < x$

$pqx > 0$ as $p > 0$, $q > 0$, $x > 0$

$\therefore f(x) - g(x) = \dfrac{(x-q)(x-p)}{pqx} < 0$