ALGEBRA (Q 1 & 2, PAPER 1)

1996

1 (a) Express $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ in the form $a\sqrt{2}-b$, where $a, b \in \mathbf{N}$.

(b) (i) $(x + 1)$ is a factor of $x^3 + 5x^2 + kx - 12$.
Find the value of k and the other two factors of the cubic expression.

(ii) If $x = \sqrt{p} + \frac{1}{\sqrt{p}} + 1$ where $p > 0$, express $x^2 - 2x$ in terms of p.

(c) (i) Make a sketch of the region of the plane represented by
$y \geq |x|$ and $y \leq 2 + |x|$.

(ii) $x^2 - px + 1$ is a factor of $ax^3 + bx + c$ where $a \neq 0$.
Show $c^2 = a(a-b)$.

SOLUTION

1 (a)

> You are often asked to rationalise the denominator of a surd. This means getting rid of surds in the denominator. Multiply above and below by the conjugate of the denominator.

$\frac{1-\sqrt{2}}{1+\sqrt{2}}$ [Multiply above and below by the conjugate.]

$= \frac{(1-\sqrt{2})}{(1+\sqrt{2})} \times \frac{(1-\sqrt{2})}{(1-\sqrt{2})}$

$= \frac{1-2\sqrt{2}+2}{1-\sqrt{2}+\sqrt{2}-2}$

$= \frac{3-2\sqrt{2}}{-1} = 2\sqrt{2}-3$

1 (b) (i) The factor theorem states that:

> If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$,
> i.e. $f(k) = 0$ and vice versa.

If $(x + 1)$ is a factor of $f(x) = x^3 + 5x^2 + kx - 12$, then $f(-1) = 0$.

$f(-1) = 0 \Rightarrow (-1)^3 + 5(-1)^2 + k(-1) - 12 = 0$

$\Rightarrow -1 + 5 - k - 12 = 0$

$\Rightarrow -8 - k = 0$

$\therefore k = -8$

To find the other two factors you can use the division process or line up the coefficients.

METHOD 1: Division process

$$\begin{array}{r|l} & x^2+4x-12 \\ \hline x+1 & x^3+5x^2-8x-12 \\ & \mp x^3 \mp x^2 \\ \hline & \quad 4x^2-8x-12 \\ & \quad \mp 4x^2 \mp 4x \\ \hline & \qquad -12x-12 \\ & \qquad \pm 12x \pm 12 \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore x^3+5x^2-8x-12=(x+1)(x^2+4x-12)$

$\Rightarrow x^3+5x^2-8x-12=(x+1)(x+6)(x-2)$

Therefore, the other two factors are $(x+6)(x-2)$.

METHOD 2: Lining up (much better, more enjoyable.)

$\therefore x^3+5x^2-8x-12=(x+1)(x^2+ax-12)$

$\Rightarrow x^3+5x^2-8x-12=x^3+ax^2-12x+x^2+ax-12$

$\Rightarrow x^3+5x^2-8x-12=x^3+(a+1)x^2+(a-12)x-12$

Lining up: $a+1=5 \Rightarrow a=4$

$\therefore x^3+5x^2-8x-12=(x+1)(x^2+4x-12)$

$\Rightarrow x^3+5x^2-8x-12=(x+1)(x+6)(x-2)$

Therefore, the other two factors are $(x+6)(x-2)$.

1 (b) (ii)

There are many ways to carry out this calculation. The longest way is to substitute the expression for x and multiply out term by term. I will show you the most efficient way.

$x^2-2x=x(x-2)$

$=\left(\sqrt{p}+\frac{1}{\sqrt{p}}+1\right)\left(\sqrt{p}+\frac{1}{\sqrt{p}}+1-2\right)$

$=\left(\left[\sqrt{p}+\frac{1}{\sqrt{p}}\right]+1\right)\left(\left[\sqrt{p}+\frac{1}{\sqrt{p}}\right]-1\right)$

Difference of 2 squares

$a^2-b^2=(a+b)(a-b)$

$=\left[\sqrt{p}+\frac{1}{\sqrt{p}}\right]^2-1$

$=p+2+\frac{1}{p}-1=p+\frac{1}{p}+1$

1 (c) (i)

If x or y is inside the modulus sign, the graph is V shaped.

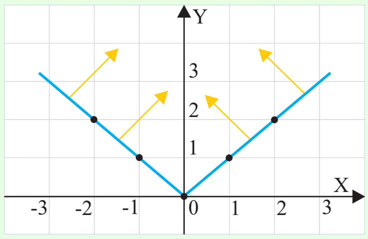
STEPS

1. Put the function inside the modulus sign equal to zero and solve for x. Use this value of x to draw up a table.
2. Choose two values to the right and two to the left of this value of x.
3. Plot the equality. It will be V-shaped.
4. Choose a point inside or outside the V (region test). Substitute it into the inequality. If it works, shade the region. If it does not work, shade the other.

Plot $y = |x|$.

$x = \mathbf{0}$

x	-2	-1	$\mathbf{0}$	1	2
$y = \lvert x \rvert$	2	1	0	1	2

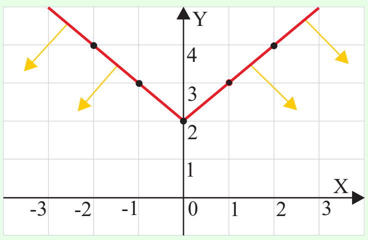


Substitute (0, 1) into the inequality $y \geq |x|$. This is true so shade the side of the line containing (0, 1).

Plot $y = 2 + |x|$.

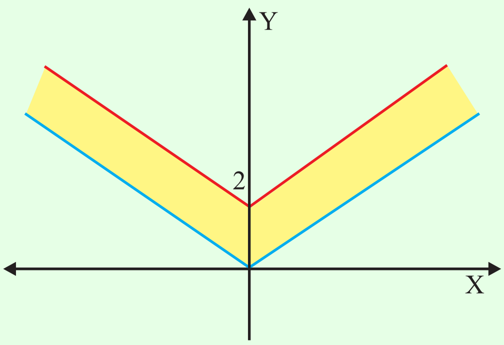
$x = \mathbf{0}$

x	-2	-1	$\mathbf{0}$	1	2
$y = 2 + \lvert x \rvert$	4	3	2	3	4



Substitute (1, 1) into the inequality $y \leq 2 + |x|$. This is true so shade the side of the line containing (1, 1).

The two regions overlap in the area between the two lines as shown.



1 (c) (ii)

Method 1: Division process

$$\begin{array}{r} ax + ap \\ x^2 - px + 1 \,\big)\, ax^3 + 0x^2 + bx + c \\ \mp ax^3 \pm apx^2 \mp ax \\ \hline apx^2 + (b-a)x + c \\ \mp apx^2 \pm ap^2x + \mp ap \\ \hline (b - a + ap^2)x + (c - ap) \end{array}$$

The remainder must be zero, i.e. $0x + 0$.

$\therefore c = ap \Rightarrow p = \frac{c}{a}$

$\therefore b - a + ap^2 = 0 \Rightarrow b - a + a(\frac{c}{a})^2 = 0$

$\Rightarrow b - a + \dfrac{c^2}{a} = 0$

$\Rightarrow ab - a^2 + c^2 = 0$

$\Rightarrow c^2 = a^2 - ab$

$\therefore c^2 = a(a-b)$

Method 2: Lining up

A cubic expression is the product of a quadratic and a linear factor.

The **first** terms of the linear and quadratics multiply to give the **first** term of the cubic. The **last** terms of the linear and quadratics multiply to give the **last** term of the cubic.

$ax^3 + 0x^2 + bx + c = (x^2 - px + 1)(ax + c)$

$\Rightarrow ax^3 + 0x^2 + bx + c = ax^3 + cx^2 - apx^2 - cpx + ax + c$

$\Rightarrow ax^3 + 0x^2 + bx + c = ax^3 + (c - ap)x^2 + (a - cp)x + c$

Lining up:

$c - ap = 0 \Rightarrow c = ap \Rightarrow p = \frac{c}{a}$

$a - cp = b \Rightarrow a - c(\frac{c}{a}) = b$

$\Rightarrow a - \dfrac{c^2}{a} = b \Rightarrow a^2 - c^2 = ab$

$\Rightarrow c^2 = a^2 - ab$

$\therefore c^2 = a(a-b)$

2 (a) Solve for x, y and z

$$x+y-z=0$$
$$x-y+z=4$$
$$x-y-z=-8$$

(b) (i) Solve for x

$$\frac{2x-7}{x+3}<1,\ x\neq 3.$$

(ii) If $u_n = n!(n+2)$ show that

$$(n+1)u_n+(n+1)!=u_{n+1}.$$

(c) Find the quadratic equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ given that $\alpha+\beta=5$ and $\alpha\beta=k$, where $k\neq 0$.
Find the range of values of k for which the equation will have real roots.

SOLUTION

2 (a)

STEPS TO SOLVING SIMULTANEOUS EQUATIONS
1. Get rid of one letter using two equations.
2. Get rid of the same letter using two other equations.
3. Solve the resulting simultaneous equation for two linears.
4. Substitute to find the other letters.

$x+y-z=0$.....**(1)**
$x-y+z=4$.....**(2)**
$x-y-z=-8$...**(3)**

Add Equations **(1)** and **(2)**: $2x=4\Rightarrow x=2$

Add equations **(2)** and **(3)**: $2x-2y=-4\Rightarrow x-y=-2$....**(4)**

Substitute the value of x into Eqn. **(4)**: $\therefore (2)-y=-2\Rightarrow y=4$

Substitute the values of x and y into Eqn. **(1)**: $(2)+(4)-z=0\Rightarrow z=6$

ANS: $x = 2$, $y = 4$, $z = 6$

2 (b) (i)

STEPS

1. Move all terms to the same side so that you have a positive coefficient of x^2.
2. Try to factorise the quadratic.
3. Solve the quadratic equation to get the roots.
4. Arrange the roots in ascending order.
5. Carry out the region test to get the solutions.

$\frac{2x-7}{x+3} < 1$ [Multiply both sides by the denominator squared.]

$\Rightarrow (x+3)(2x-7) < (x+3)^2$

$\Rightarrow (x+3)(2x-7) - (x+3)^2 < 0$ [Factorise the left-hand side]

$\Rightarrow (x+3)[2x-7-x-3] < 0$

$\Rightarrow (x+3)(x-10) < 0$

Solve the equality: $(x+3)(x-10) = 0 \Rightarrow x = -3, 10$

Do the region test:

←	-3	↔	10	→
-4		0		11
$(-1)(-14) < 0$		$(3)(-10) < 0$		$(14)(1) < 0$
Wrong		Correct		Wrong

Region Test on $(x+3)(x-10) < 0$ .....**Test Box**

$\therefore -3 < x < 10$

2 (b) (ii)

$u_n = n!(n+2)$

$\Rightarrow u_{n+1} = (n+1)!(n+3)$

LHS	***RHS***
$(n+1)u_n + (n+1)!$	u_{n+1}
$= (n+1)n!(n+2) + (n+1)!$	$= (n+1)!(n+3)$
$= (n+2)! + (n+1)!$	
$= (n+1)!(n+2+1)$	
$= (n+1)!(n+3)$	

2 (c)

Quadratic Equation: $x^2 - Sx + P = 0$

Roots: $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum S: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{5}{k}$

Product P: $\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{k}$

Forming a quadratic equation given its roots:

$x^2 - \mathbf{S}x + \mathbf{P} = 0$

Sum S: $\alpha + \beta = -\frac{b}{a} = \frac{-2^{\text{nd.}}}{1^{\text{st.}}}$

Product P: $\alpha\beta = \frac{c}{a} = \frac{3^{\text{rd.}}}{1^{\text{st.}}}$

Quadratic: $x^2 - \frac{5}{k}x + \frac{1}{k} = 0$

$\Rightarrow kx^2 - 5x + 1 = 0$

Remember: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.

If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

$a = k$

$b = -5$

$c = 1$

$b^2 - 4ac \geq 0 \Rightarrow (-5)^2 - 4k(1) \geq 0$

$\Rightarrow 25 - 4k \geq 0$

$\Rightarrow 25 \geq 4k$

$\Rightarrow \frac{25}{4} \geq k$

$\therefore k \leq \frac{25}{4}$