SEQUENCES & SERIES (Q 5, PAPER 1)

LESSON NO. 8: GEOMETRIC SERIES

2007

5 (c) The first two terms of a geometric series are $\frac{1}{3}+\frac{1}{9}+...$

(i) Find r, the common ratio.

(ii) Find an expression for S_n, the sum of the first n terms.

Write your answer in the form $\frac{1}{k}\left(1-\frac{1}{3^n}\right)$ where $k \in \mathbf{N}$.

(iii) The sum of the first n terms of the geometric series $\frac{p}{3}+\frac{p}{9}+....$ is $1-\frac{1}{3^n}$.

Find the value of p.

SOLUTION

5 (c)

Geometric series: $\frac{1}{3}+\frac{1}{9}+....$

5 (c) (i)

$$r=\frac{\frac{1}{9}}{\frac{1}{3}}=\frac{1}{9}\times\frac{3}{1}=\frac{1}{3}$$

Any term $\div$ Previous term $=\frac{T_n}{T_{n-1}}=$ Constant (r)

5 (c) (ii)

$a=\frac{1}{3}$

$r=\frac{1}{3}$

Summing formula: $S_n=\frac{a(1-r^n)}{(1-r)}$ 5

$$\therefore S_n=\frac{a(1-r^n)}{1-r}=\frac{(\frac{1}{3})(1-(\frac{1}{3})^n)}{1-(\frac{1}{3})}$$

$$=\frac{(\frac{1}{3})(1-(\frac{1}{3})^n)}{\frac{2}{3}}$$

$$=\frac{1}{2}\left(1-\frac{1}{3^n}\right)$$

5 (c) (iii)

Geometric series: $\frac{p}{3}+\frac{p}{9}+...$

$a=\frac{p}{3}$

$$r=\frac{\frac{p}{9}}{\frac{p}{3}}=\frac{1}{3}$$

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{(\frac{p}{3})(1-(\frac{1}{3})^n)}{1-\frac{1}{3}}$$

$$=\frac{(\frac{p}{3})(1-(\frac{1}{3})^n)}{\frac{2}{3}}$$

$$=(\tfrac{p}{2})(1-(\tfrac{1}{3})^n)$$

$$=\frac{p}{2}\left(1-\frac{1}{3^n}\right)$$

CONT...

You are told that the sum of the first n terms is given by $1-\frac{1}{3^n}$.

$$\therefore \frac{p}{2}\left(1-\frac{1}{3^n}\right)=1-\frac{1}{3^n}$$

$$\Rightarrow \frac{p}{2}=1$$

$$\Rightarrow p=2$$

2006

5 (b) The nth term of a geometric series is

$T_n = 4(\frac{1}{2})^n$.

(i) Find a, the first term.

(ii) Find r, the common ratio.

(iii) Write $4-S_{10}$ in the form $\frac{1}{2^k}$, $k \in \mathbf{N}$, where S_{10} is the sum of the first ten terms.

SOLUTION

5 (b) (i)

Replace n by 1 in the general term to find a.

$$a = T_1$$

$$T_n = 4(\tfrac{1}{2})^n$$

$$\Rightarrow T_1 = 4(\tfrac{1}{2})^1 = 4(\tfrac{1}{2}) = 2$$

5 (b) (ii)

To find the common ratio, r, find the second term, T_2, and then divide the second term by the first term.

$$T_n = 4(\tfrac{1}{2})^n$$

$$\Rightarrow T_2 = 4(\tfrac{1}{2})^2 = 4(\tfrac{1}{4}) = 1$$

$$\therefore r = \frac{T_2}{T_1} = \tfrac{1}{2}$$

CONT...

5 (b) (iii)

$n = 10$

$a = 2$

$r = \frac{1}{2}$

Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$ 5

$$\therefore S_{10} = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_{10} = \frac{2(1-(\frac{1}{2})^{10})}{(1-\frac{1}{2})} = \frac{2(1-(\frac{1}{2})^{10})}{\frac{1}{2}}$$

$$\Rightarrow S_{10} = 4(1-(\tfrac{1}{2})^{10}) = 4 - 4(\tfrac{1}{2})^{10}$$

$$\therefore 4 - S_{10} = 4 - 4 + 4(\tfrac{1}{2})^{10}$$

$$= 4(\tfrac{1}{2})^{10} = 2^2 \times \frac{1}{2^{10}} = \frac{2^2}{2^{10}}$$

$$= \frac{1}{2^8}$$

2004

5 (c) The first term of a geometric series is 1 and the common ratio is -4.

(i) Write down the first three terms of the series.

(ii) Find S_6, the sum of the first 6 terms.

(iii) Show that $16S_4 - 3 = S_6$, where S_4 is the sum of the first 4 terms.

SOLUTION

5 (c) (i)

Write down the first term and keep on multiplying by the common ratio.

$a = 1,\ r = -4$

Geometric series: $1,\ -4,\ 16, ..$

5 (c) (ii)

Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$ 5

$n = 6,\ a = 1,\ r = -4$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_6 = \frac{1(1-(-4)^6)}{(1-(-4))}$$

$$\Rightarrow S_6 = \frac{(1-4096)}{5}$$

$$\Rightarrow S_6 = \frac{-4095}{5} = -819$$

CONT...

5 (c) (iii)

$n = 4,\ a = 1,\ r = -4$

$$16S_4 - 3 = 16\left(\frac{1(1-(-4)^4)}{(1-(-4))}\right) - 3$$

$$= 16\left(\frac{(1-256)}{5}\right) - 3$$

$$= 16\left(\frac{-255}{5}\right) - 3$$

$$= 16(-51) - 3$$

$$= -816 - 3 = -819$$

2003

5 (b) The first two terms of a geometric series are 32 + 8 +...

(i) What us the value of r, the common ratio?

(ii) Find an expression for S_n, the sum of the first n terms.

(iii) Find S_{10}, the sum of the first 10 terms.
Given your answer correct to four decimal places.

SOLUTION

5 (b) (i)

Geometric series: 32 + 8 + ...

r = Common ratio = Any term ÷ Previous term

$$\therefore r = \tfrac{8}{32} = \tfrac{1}{4}$$

5 (b) (ii)

Summing formula: $\boxed{S_n = \frac{a(1-r^n)}{(1-r)}}$ 5

$a = 32,\ r = \tfrac{1}{4}$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_n = \frac{32(1-(\frac{1}{4})^n)}{(1-\frac{1}{4})}$$

$$\Rightarrow S_n = \frac{32(1-(\frac{1}{4})^n)}{\frac{3}{4}}$$

$$\Rightarrow S_n = \tfrac{128}{3}(1-(\tfrac{1}{4})^n)$$

5 (b) (iii)

$S_{10} = \tfrac{128}{3}(1-(\tfrac{1}{4})^{10}) = 42.6666$ [Use calculator]

2001

5 (b) The nth term of a geometric series is given by $T_n = 3^n$.

(i) What is the value of a, the first term?

(ii) What is the value of r, the common ratio?

(iii) Show that S_{10}, the sum of the first ten terms, is $\frac{3}{2}(3^{10}-1)$.

SOLUTION

5 (b) (i)

$T_n = 3^n$

$\therefore T_1 = a = 3^1 = 3$

5 (b) (ii)

r = Common ratio = Any term ÷ Previous term

Find the second term, T_2, and then find the common ratio r by dividing the second term by the first term.

$T_2 = 3^2 = 9$

$\therefore r = \frac{9}{3} = 3$

5 (b) (iii)

Summing formula: $\boxed{S_n = \frac{a(1-r^n)}{(1-r)}}$ **5**

$a = 3,\ r = 3,\ n = 10$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_{10} = \frac{3(1-3^{10})}{(1-3)}$$

$$\Rightarrow S_{10} = \frac{3(1-3^{10})}{-2}$$

$$\Rightarrow S_{10} = -\tfrac{3}{2}(1-3^{10})$$

$$\Rightarrow S_{10} = \tfrac{3}{2}(3^{10}-1)$$

2000

5 (b) The first term of a geometric series is 1 and the common ratio is $\frac{11}{10}$.

(i) Write down the second, thirds and fourth terms of the series.

(ii) Calculate S_4, the sum of the first four terms. Give your answer as a decimal.

SOLUTION

5 (b) (i)

To generate the terms of a geometric series multiply each term by the ratio to get the next term.

$a = T_1 = 1$

$T_2 = 1 \times \frac{11}{10} = \frac{11}{10}$

$T_3 = \frac{11}{10} \times \frac{11}{10} = \frac{121}{100}$

$T_4 = \frac{121}{100} \times \frac{11}{10} = \frac{1331}{1000}$

5 (b) (ii)

$S_4 = T_1 + T_2 + T_3 + T_4 = 1 + \frac{11}{10} + \frac{121}{100} + \frac{1331}{1000} = 1 + 1.1 + 1.21 + 1.331 = 1.641$

1999

5 (b) The first two terms of a geometric series are $2 + \frac{2}{3} + \ldots$

(i) Find r, the common ratio.

(ii) Write down the third and fourth terms of the series.

(iii) Show that S_6, the sum to 6 terms, is $3 - \dfrac{1}{3^5}$.

SOLUTION

5 (b) (i)

$\boxed{r = \text{Common ratio} = \text{Any term} \div \text{Previous term}}$

$r = \dfrac{\frac{2}{3}}{2} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

5 (b) (ii)

To generate the terms of a geometric sequence, keep on multiplying each term by the common ratio r to get the next term.

$T_3 = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

$T_4 = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$

CONT...

5 (b) (iii)

$a = 2,\ r = \frac{1}{3},\ n = 6$ Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$ **5**

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_6 = \frac{2(1-(\frac{1}{3})^6)}{(1-(\frac{1}{3}))}$$

$$\Rightarrow S_6 = \frac{2(1-(\frac{1}{3})^6)}{(1-(\frac{1}{3}))}$$

$$\Rightarrow S_6 = \frac{2(1-(\frac{1}{3})^6)}{\frac{2}{3}} \quad [\textbf{Note: } \frac{2}{\frac{2}{3}} = 2\times\tfrac{3}{2} = 3\,]$$

$$\Rightarrow S_6 = 3(1-(\tfrac{1}{3})^6)$$

$$\Rightarrow S_6 = 3 - 3(\tfrac{1}{3})^6 \quad [\textbf{Note: } 3(\tfrac{1}{3})^6 = 3^1 \times \frac{1}{3^6} = \frac{1}{3^5}\,]$$

$$\Rightarrow S_6 = 3 - \frac{1}{3^5}$$

1998

5 (b) The nth term of a geometric sequence is

$$T_n = \frac{2^n}{3^n}.$$

(i) Find the first three terms of the sequence.

(ii) Show that S_5, the sum of the first five terms, is $\frac{422}{243}$.

Solution

5 (b) (i)

$$T_n = \frac{2^n}{3^n}$$

$$\therefore T_1 = \frac{2^{(1)}}{3^{(1)}} = \frac{2}{3}$$

$$\therefore T_2 = \frac{2^{(2)}}{3^{(2)}} = \frac{4}{9}$$

$$\therefore T_3 = \frac{2^{(3)}}{3^{(3)}} = \frac{8}{27}$$

Geometric sequence: $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

Cont...

5 (b) (ii)

Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$ 5

$\boxed{r = \text{Common ratio} = \text{Any term} \div \text{Previous term}}$

$r = \dfrac{\frac{4}{9}}{\frac{2}{3}} = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$

$a = \frac{2}{3},\ r = \frac{2}{3},\ n = 5$

$S_n = \dfrac{a(1-r^n)}{(1-r)}$

$\Rightarrow S_5 = \dfrac{(\frac{2}{3})(1-(\frac{2}{3})^5)}{(1-(\frac{2}{3}))}$

$\Rightarrow S_5 = \dfrac{(\frac{2}{3})(1-\frac{32}{243})}{\frac{1}{3}}$

$\Rightarrow S_5 = 2(\frac{211}{243}) = \frac{422}{243}$

1996

5 (b) The nth term, T_n, of a geometric series is

$$T_n = 3^{n-1}.$$

Find

(i) T_1, the first term

(ii) r, the common ratio

(iii) S_n, the sum to n terms.

Investigate if

$$2S_n - T_n = 2T_n - 1.$$

Solution

5 (b) (i)

$T_n = 3^{n-1} \Rightarrow T_1 = 3^{(1)-1} = 3^0 = 1$

5 (b) (ii)

$\boxed{r = \text{Common ratio} = \text{Any term} \div \text{Previous term}}$

Find the second term T_2 and then divide the second term by the first term to find r.

$T_n = 3^{n-1} \Rightarrow T_2 = 3^{(2)-1} = 3^1 = 3$

$\Rightarrow r = \dfrac{T_2}{T_1} = \dfrac{3}{1} = 3$

Cont...

5 (b) (iii)

$a = 1,\ r = 3$

Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$ 5

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\Rightarrow S_n = \frac{1(1-3^n)}{(1-3)}$$

$$\Rightarrow S_n = \frac{1(1-3^n)}{-2}$$

$$\Rightarrow S_n = \tfrac{1}{2}(3^n - 1)$$

LHS	*RHS*
$2S_n - T_n$	$2T_n - 1$
$= 2[\tfrac{1}{2}(3^n - 1)] - 3^{n-1}$	$= 2(3^{n-1}) - 1$
$= 3^n - 1 - 3^{n-1}$	
$= (3^n - 3^{n-1}) - 1$	
$= 3^{n-1}(3^1 - 1) - 1$	
$= 3^{n-1}(2) - 1$	

$$\therefore 2S_n - T_n = 2T_n - 1$$