GEOMETRY (Q 4, PAPER 2)

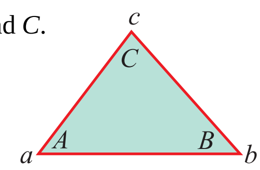
LESSON NO. 4: THE TEN THEOREMS

This section will be done in a different way to previous solutions. Thev ten theorems are listed with the years in which they were asked listed at the top.

THEOREM 1 (LC 2003 4 (b))

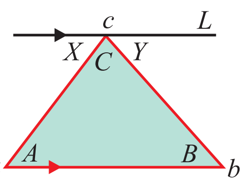
THEOREM 1: The sum of the degree measures of the interior angles of a triangle is 180°.

GIVEN: Δabc with angles A, B and C.



TO PROVE: $A+B+C=180^\circ$

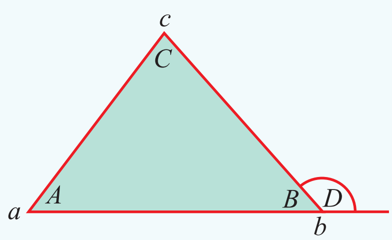
CONSTRUCTION: Draw a line L through c parallel to the base of the triangle and label angles X and Y as shown.



PROOF: $X+Y+C=180^\circ$ [Straight angle]
$X=A$ [Alternate angles]
$Y=B$ [Alternate angles]
$\therefore A+B+C=180^\circ$

DEDUCTION 1: The degree measure of the exterior angle of a triangle is equal to the sum of the two remote interior angles.

GIVEN: Δabc with angles A, B and C and exterior angle D.

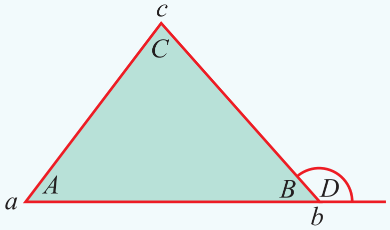


TO PROVE: $D = A + C$

PROOF: $A + B + C = 180^\circ$ [Theorem 1]
$B + D = 180^\circ$ [Straight angle]
Therefore, $B + D = A + B + C$
$\therefore D = A + C$.

Deduction 2: An exterior angle of a triangle is greater than either of the two remote (opposite) interior angles.

Given: Δabc with angles A, B and C and exterior angle D.



To prove: $D > A$ and $D > C$.

Proof: $D = A + C$. Since $C > 0$, $D > A$.
$D = A + C$. Since $A > 0$, $D > C$.

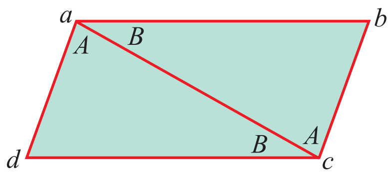
Theorem 2 (LC 2004 4 (b))

Theorem 2: The opposite sides of a parallelogram have equal lengths.

Given: Parallelogram *abcd*.

To Prove: $|ab| = |dc|$ and $|ad| = |bc|$.

Construction: Join *a* to *c*.



Proof: Δabc and Δadc are congruent (**ASA**) because:

$|\angle acb| = |\angle dac| = A$ [Alternate angles]

$|\angle bac| = |\angle dca| = B$ [Alternate angles]

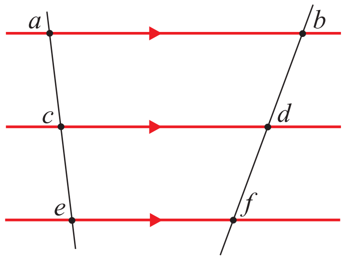
$|ac| = |ac|$ [Common side]

Therefore, $|ab| = |dc|$ and $|ad| = |bc|$.

Theorem 3 (LC 2002 4 (b))

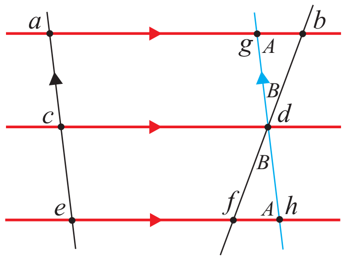
Theorem 3: If three parallel lines make intercepts of equal length on a transversal, then they will make intercepts of equal lengths on any other transversal.

Given: Three parallel lines *ab*, *cd* and *ef* such that *c* is on *ae* and *d* is on *bf* with $|ac| = |ce|$.



To Prove: $|bd| = |df|$.

Construction: Draw a line *gh* through *d* parallel to *ae* such that *g* is on *ab* and *h* is on *ef*.



Proof: *acdg* is a parallelogram $\Rightarrow |ac| = |gd| = |ce|$

cehd is a parallelogram $\Rightarrow |ce| = |dh|$.

$\therefore |gd| = |dh|$.

Now Δgdb and Δfdh are congruent (**ASA**) because:

$|\angle bgd| = |\angle fhd| = A$ [Alternate angles]

$|\angle gdb| = |\angle fdh| = B$ [Vertically opposite angles]

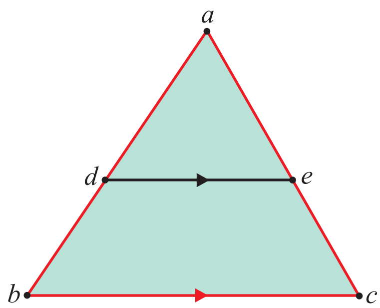
$|gd| = |dh|$ [Already proved]

$\therefore |bd| = |df|$.

THEOREM 4 (LC 2005, 2001, 1996 4 (b))

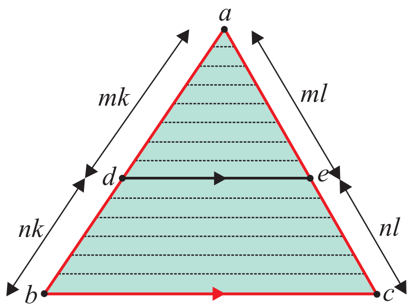
THEOREM 4: A line which is parallel to one side of a triangle, and cuts a second side, will cut the third side in the same proportion as the second.

GIVEN: Δabc and line de parallel to bc which cuts $[ab]$ in the ratio m:n.



TO PROVE: $\dfrac{|ad|}{|db|}=\dfrac{|ae|}{|ec|}=\dfrac{m}{n}$

CONSTRUCTION: Divide $[ab]$ into m and n parts each of length k so that $|ad| = mk$ and $|db| = nk$.



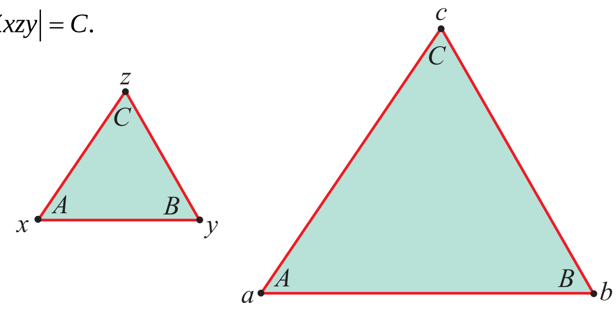
PROOF: According to theorem 3, $[ac]$ is also divided into $m + n$ equal parts each of length l.

$$\frac{|ae|}{|ec|}=\frac{ml}{nl}=\frac{m}{n}=\frac{|ad|}{|db|}.$$

THEOREM 5

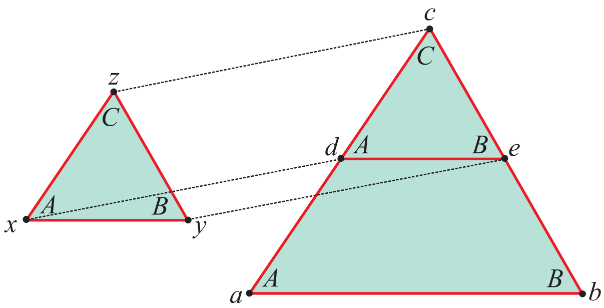
THEOREM 5: If the three angles of one triangle have a degree measure equal respectively to the degree measure of the angles of a second triangle then the lengths of the corresponding sides of the two triangles are proportional.

GIVEN: Δabc and Δxyz with $|\angle cab| = |\angle zxy| = A,$ $|\angle abc| = |\angle xyz| = B$ and $|\angle acb| = |\angle xzy| = C.$



TO PROVE: $\dfrac{|cb|}{|zy|} = \dfrac{|ca|}{|zx|} = \dfrac{|ab|}{|xy|}.$

CONSTRUCTION: Translate Δxyz so that $z \to c,$ $x \to d$ and $y \to e.$



PROOF: $[de]$ is parallel to $[ab]$ because $|\angle cde| = |\angle cab| = A$ and

$|\angle ced| = |\angle cba| = B.$ [Corresponding angles]

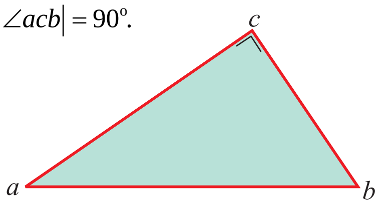
$\therefore \dfrac{|cb|}{|ce|} = \dfrac{|ca|}{|cd|} \Rightarrow \dfrac{|cb|}{|zy|} = \dfrac{|ca|}{|zx|}.$ [Theorem 4]

Similarly $\dfrac{|ca|}{|zx|} = \dfrac{|ab|}{|xy|}.$

THEOREM 6 (LC 2000 4 (b))

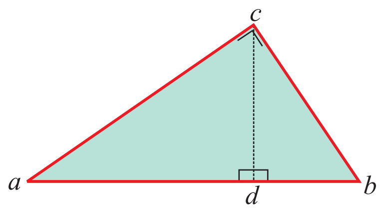
THEOREM 6: In a right-angled triangle the square of the length of the side opposite to the right-angle is equal to the sum of the squares of the lengths of the other two sides.

GIVEN: Δabc with $|\angle acb| = 90°$.

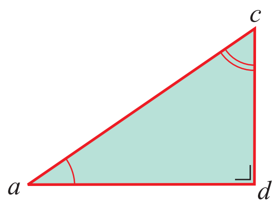


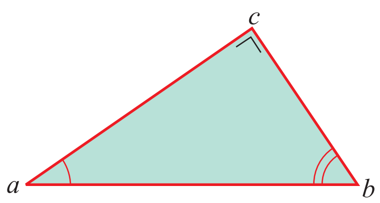
TO PROVE: $|ab|^2 = |ac|^2 + |bc|^2$.

CONSTRUCTION: Draw a perpendicular [cd] onto [ab].



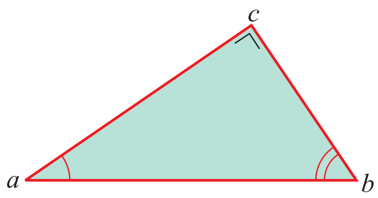
PROOF: Δadc and Δabc are equiangular.

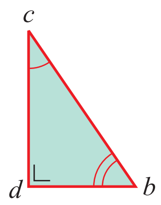




$$\therefore \frac{|ab|}{|ac|} = \frac{|ac|}{|ad|} \Rightarrow |ac|^2 = |ab||ad| \text{.....}\mathbf{1}$$

Δabc and Δdbc are equiangular.





$$\therefore \frac{|bc|}{|ab|} = \frac{|db|}{|bc|} \Rightarrow |bc|^2 = |ab||db| \text{.....}\mathbf{2}$$

Adding equation **1** and **2**:

$$\Rightarrow |ac|^2 + |bc|^2 = |ab||ad| + |ab||db|$$

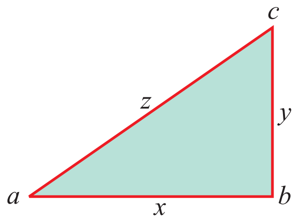
$$\Rightarrow |ac|^2 + |bc|^2 = |ab|\{|ad| + |db|\} = |ab||ab| = |ab|^2$$

$$\therefore |ac|^2 + |bc|^2 = |ab|^2$$

Theorem 7

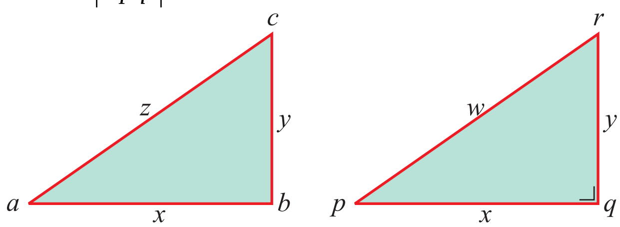
Theorem 7: (Converse of Pythagoras) If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides then the triangle has a right angle and this is opposite the longest side.

Given: Δabc with $|ab| = x$, $|bc| = y$, $|ac| = z$ and $z^2 = x^2 + y^2$.



To Prove: $|\angle abc| = 90^{\circ}$.

Construction: Construct another Δpqr with $|pq| = x$, $|qr| = y$ and $|pr| = w$ and $|\angle pqr| = 90^{\circ}$.



Proof: By Pythagoras on $\Delta pqr \Rightarrow w^2 = x^2 + y^2$

But $\Rightarrow z^2 = x^2 + y^2$ [Given]

$\therefore w = z$

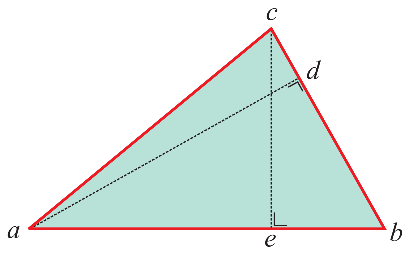
$\therefore \Delta abc$ and Δpqr are congruent (**SSS**).

$\therefore |\angle abc| = |\angle pqr| = 90^{\circ}$.

Theorem 8 (LC 2007, 1997 4 (b))

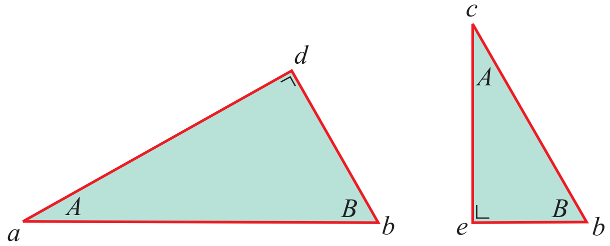
Theorem 8: The products of the lengths of the sides of a triangle by the corresponding altitudes are equal.

Given: Δabc with altitudes $[ad]$ and $[ce]$.



To Prove: $|ad||bc| = |ab||ce|$.

Construction: Separate out Δabd and Δebc.



Proof: Δabd and Δebc are equiangular because:

$|\angle adb| = |\angle ceb| = 90^\circ$

$|\angle abd| = |\angle ebc| = B$ [Common]

$|\angle dab| = |\angle ecb| = A$ [Remaining angle]

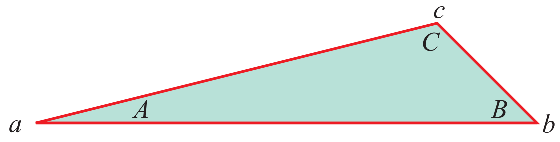
$$\therefore \frac{|ad|}{|ce|} = \frac{|ab|}{|bc|} \Rightarrow |ad||bc| = |ab||ce|$$

Note: An **altitude** is a perpendicular line from a vertex of a triangle to the opposite side.

Theorem 9 (LC 2006, 1998 4 (b))

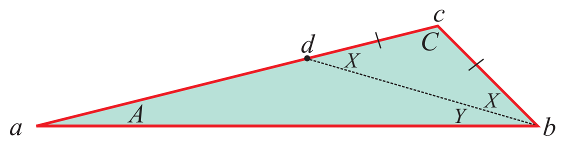
Theorem 9: If the lengths of a triangle are unequal, then the degree measures of the angles opposite to them are unequal, with the greater angle opposite to the longer side.

Given: Δabc with $|ac| > |bc|$.



To Prove: $|\angle abc| > |\angle cab|$.

Construction: Mark a point d on $[ac]$ such that $|cd| = |cb|$.



Proof: $|dc| = |cb| \Rightarrow |\angle cdb| = |\angle dbc| = X$ [Isosceles triangle]

$B = |\angle abc| = X + Y$

$X = A + Y$ [Exterior Angle]

$\Rightarrow A = X - Y$

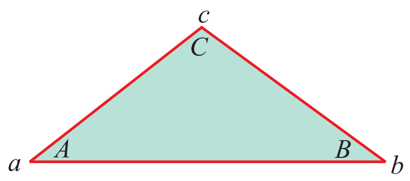
$\therefore B > A \Rightarrow |\angle abc| = |\angle cab|$.

Similarly, this is true for the other sides.

Theorem 10 (LC 1999 4 (b))

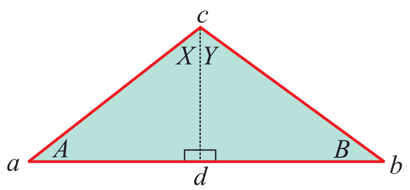
Theorem 10: The sum of the lengths of any two sides of a triangle is greater than that of the third side.

Given: Δabc



To Prove: $|ab| < |ac| + |bc|$

Construction: Draw a perpendicular $[cd]$ onto $[ab]$.



Proof: $X < 90^\circ \Rightarrow |ad| < |ac|$......**1**

$Y < 90^\circ \Rightarrow |db| < |bc|$......**2**

Adding **1** and **2**: $|ad| + |db| < |ac| + |bc|$

$\therefore |ab| < |ac| + |bc|$.