DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

LESSON NO. 14: RECIPROCAL GRAPHS

2007

8 (c) Let $f(x)=\dfrac{1}{x+7}$, $x\in\mathbf{R}$, $x\neq -7$.

(i) Given that $f(k) = 1$, find k.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Show that the curve $y = f(x)$ has no turning points.

SOLUTION

8 (c) (i)

$$f(x)=\frac{1}{x+7}$$

$f(k)=1\Rightarrow \dfrac{1}{(k+7)}=1$ [Multiply across by $(k + 7)$.]

$\Rightarrow k+7=1$

$\Rightarrow k=-6$

8 (c) (ii)

$$f(x)=\frac{1}{(x+7)}=(x+7)^{-1}$$

$\Rightarrow f'(x)=-1(x+7)^{-2}(1)$

$\Rightarrow f'(x)=-\dfrac{1}{(x+7)^2}$

$y=(u)^n \Rightarrow \dfrac{dy}{dx}=n(u)^{n-1}\times\dfrac{du}{dx}$ ... 1

REMEMBER IT AS:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

8 (c) (iii)

$f'(x)=0$

$\Rightarrow -\dfrac{1}{(x+7)^2}=0$ [Multiply across by $(x+7)^2$.]

$\Rightarrow -1=0$ [This equation is nonsense and has no solutions.]

Turning Point $\Rightarrow \dfrac{dy}{dx}=0$ 6

To find the turning points set $\dfrac{dy}{dx}=0$ and solve for x.

As there are no solutions for $f'(x)=0$, the curve $y = f(x)$ has no turning points.

2006

8 (c) Let $f(x)=\dfrac{1}{x-2}$, $x\in\mathbf{R}$, $x\neq 2$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the values of x for which $f'(x)=-1$.

(iii) Find the co-ordinates of the two points on the curve $y = f(x)$ at which the slope of the tangent is -1.

Solution

8 (c) (i)

$$f(x)=\frac{1}{x-2}=(x-2)^{-1}$$

$$\Rightarrow f'(x)=-1(x-2)^{-2}(1)=-\frac{1}{(x-2)^2}$$

$$\boxed{y=(u)^n\Rightarrow\frac{dy}{dx}=n(u)^{n-1}\times\frac{du}{dx}}\quad\ldots 1$$

Remember it as:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

8 (c) (ii)

Put $f'(x)=-1$ and solve for x.

$$f'(x)=-1\Rightarrow-\frac{1}{(x-2)^2}=-1$$

$$\Rightarrow 1=1(x-2)^2$$

$$\Rightarrow \pm1=(x-2)$$

$$\therefore 1=x-2\Rightarrow x=3$$

$$\therefore -1=x-2\Rightarrow x=1$$

Power Rules

4. $a^{-n}=\dfrac{1}{a^n}$ **Ex.** $x^{-3}=\dfrac{1}{x^3}$

8 (c) (iii)

You have already found the values of x for which the slope of the tangents are -1. To find their corresponding y values (or $f(x)$ values) substitute these values of x into the original equation.

$$x=1:\ f(x)=\frac{1}{x-2}\Rightarrow f(1)=\frac{1}{(1)-2}=\frac{1}{-1}=-1$$

$$x=3:\ f(x)=\frac{1}{x-2}\Rightarrow f(3)=\frac{1}{(3)-2}=\frac{1}{1}=1$$

The co-ordinates of the points are: $(1,-1)$, $(3, 1)$.

2004

8 (c) Let $f(x)=\dfrac{1}{x+3}$, $x\in \mathbf{R}$, $x\neq -3$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) There are two points on the curve $y = f(x)$ at which the slope of the tangent is -1. Find the co-ordinates of these two points.

(iii) Show that no tangent to the curve $y = f(x)$ has a slope of 1.

SOLUTION

8 (c) (i)

$$f(x)=\frac{1}{x+3}=(x+3)^{-1}$$

$$\Rightarrow f'(x)=-1(x+3)^{-2}(1)=-\frac{1}{(x+3)^2}$$

$y=(u)^n \Rightarrow \dfrac{dy}{dx}=n(u)^{n-1}\times\dfrac{du}{dx}$ ... 1

REMEMBER IT AS:

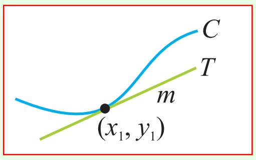
Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

POWER RULES

4. $a^{-n}=\dfrac{1}{a^n}$ **Ex.** $x^{-3}=\dfrac{1}{x^3}$

8 (c) (ii)

GOING BACKWARDS: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.



STEPS

1. Differentiate the equation of the curve: $\dfrac{dy}{dx}$.

2. Put $\dfrac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.

3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

1. $f'(x)=\dfrac{dy}{dx}=-\dfrac{1}{(x+3)^2}$

2. $-\dfrac{1}{(x+3)^2}=-1\Rightarrow 1=(x+3)^2$

$\Rightarrow x+3=\pm 1$

$\therefore x=-2,\ -4$

3. $x=-2: y=f(-2)=\dfrac{1}{(-2)+3}=\dfrac{1}{1}=1\Rightarrow(-2,\ 1)$ is a points of contact with the tangent.

$x=-4: y=f(-4)=\dfrac{1}{(-4)+3}=\dfrac{1}{-1}=-1\Rightarrow(-4,\ -1)$ is a points of contact with the tangent.

CONT....

8 (c) (iii)

Put $\frac{dy}{dx}=1$ and show that there exists no solutions for x.

$\frac{dy}{dx}=1 \Rightarrow -\frac{1}{(x+3)^2}=1$

$\Rightarrow -1=(x+3)^2$

$\Rightarrow \sqrt{-1}=(x+3)$

$\sqrt{-1}$ has no real solutions. Therefore, no tangent to the curve has a slope of 1.

2002

8 Let $f(x)=\frac{1}{x+2}$.

(i) Find $f(-6)$, $f(-3)$, $f(-1)$, $f(0)$ and $f(2)$.

(ii) For what real value of x is $f(x)$ not defined?

(iii) Draw the graph of $f(x)=\frac{1}{x+2}$ for $-6 \le x \le 2$.

(iv) Find $f'(x)$, the derivative of $f(x)$.

(v) Find the two values of x at which the slope of the tangent to the graph is $-\frac{1}{9}$.

(vi) Show that there is no tangent to the graph of f that is parallel to the x-axis.

SOLUTION

DRAWING RECIPROCAL GRAPHS

STEPS

1. Find the gap first by putting the bottom of the function equal to zero and solving for x.
2. Find other values of $f(x)$ by putting in values of x as given in the domain.
3. Plot these points by joining up smoothly and continuing towards the vertical line but never touching it.

CONT....

8 (i)

$$f(x)=\frac{1}{x+2}$$

$$f(-6)=\frac{1}{-6+2}=-\frac{1}{4}=-0.25$$

$$f(-3)=\frac{1}{-3+2}=-\frac{1}{1}=-1$$

$$f(-1)=\frac{1}{-1+2}=\frac{1}{1}=1$$

$$f(0)=\frac{1}{0+2}=\frac{1}{2}=0.5$$

$$f(2)=\frac{1}{2+2}=\frac{1}{4}=0.25$$

8 (ii)

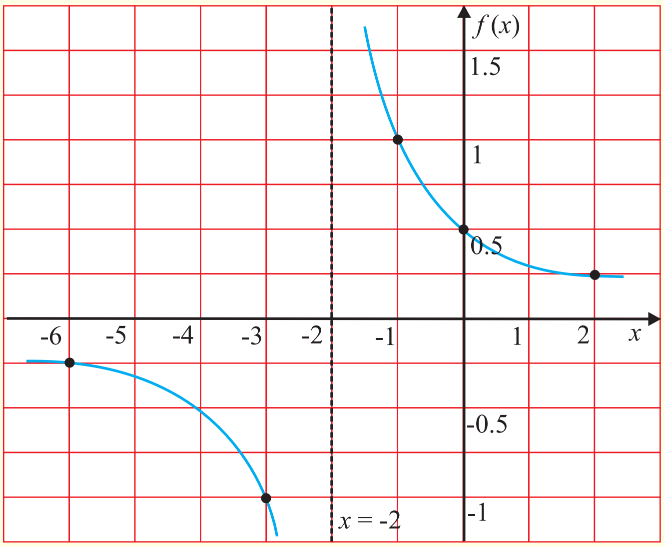
Put the bottom equal to zero. This will give the equation of the gap or the asymptote in the graph. It is also the value of x for which $f(x)$ is not defined.

$$x+2=0 \Rightarrow x=-2$$

8 (iii)

Use the values from part **(i)** and the asymptote equation in part **(ii)** to draw the graph.

Points: $(-6, 0.25)$, $(-3, -1)$, $(-1, 1)$, $(0, 0.5)$, $(2, 0.25)$



8 (iv)

$$f(x)=\frac{1}{(x+2)}=(x+2)^{-1}$$

$$\Rightarrow f'(x)=-1(x+2)^{-2}(1)=-\frac{1}{(x+2)^2}$$

CONT....

8 (v)

STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

1. $f'(x)=\frac{dy}{dx}=-\frac{1}{(x+2)^2}$

2. $-\frac{1}{(x+2)^2}=-\frac{1}{9}\Rightarrow 9=(x+2)^2$ [Multiply across by $9(x+2)^2$.]

$\Rightarrow \pm 3=x+2$

$\therefore x=-5, 1$

Step **3** is not required as you need to find the x values only.

8 (vi)

Any line parallel to the x-axis has a slope of zero.

Put $\frac{dy}{dx}$ equal to zero and show it has no solutions.

$\frac{dy}{dx}=0\Rightarrow -\frac{1}{(x+2)^2}=0$ [Multiply across by $(x+2)^2$.]

$\Rightarrow -1=0$

This equation is nonsense. Therefore, there are no tangents to the graph that are parallel to the x-axis.

2001

8 (c) Let $f(x)=\frac{1}{x+1}$ for $x\in\mathbf{R}$ and $x>-1$.

(i) Find $f'(x)$.

(ii) Find the co-ordinates of the point on the curve of $f(x)$ at which the tangent has slope of $-\frac{1}{4}$.

(iii) Find the equation of the tangent to the curve which has slope of $-\frac{1}{4}$.

SOLUTION

8 (c) (i)

$f(x)=\frac{1}{(x+1)}=(x+1)^{-1}$

$\Rightarrow f'(x)=-1(x+1)^{-2}(1)=-\frac{1}{(x+1)^2}$

CONT....

8 (c) (ii)

Finding the coordinates of the point of contact of the tangent given the slope:

STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

1. $y = f(x) = \dfrac{1}{(x+1)}$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -\frac{1}{(x+1)^2}$$

2. $\dfrac{dy}{dx} = 0 \Rightarrow -\dfrac{1}{(x+1)^2} = -\dfrac{1}{4}$

$\Rightarrow (x+1)^2 = 4$

$\Rightarrow (x+1) = \pm 2$

$\therefore x = 1, -3$ [As $x > -1$ ignore one of the solutions.]

$\therefore x = 1$

3. $x = 1$: $y = f(1) = \dfrac{1}{(1+1)} = \frac{1}{2} \Rightarrow (1, \frac{1}{2})$ is the point of contact.

8 (c) (iii)

Equation of a line: $\boxed{y - y_1 = m(x - x_1)}$ **4**

(x_1, y_1) is a point on the line and m is the slope of the line.

Point $(1, \frac{1}{2})$, $m = -\frac{1}{4}$

$y - \frac{1}{2} = -\frac{1}{4}(x-1)$

$\Rightarrow 4(y - \frac{1}{2}) = -1(x-1)$

$\Rightarrow 4y - 2 = -1x + 1$

$\Rightarrow x + 4y - 3 = 0$

1998

8 Let $f(x)=\dfrac{1}{x-1}$, for $x\in\mathbf{R}$ and $x\neq 1$.

(i) Find the value of $f(-2)$, $f(0)$, $f(\frac{3}{2})$ and $f(5)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Draw the graph of $f(x)$ for $-2\leq x\leq 5$.

(iv) Find the equation of the tangent T to the curve at the point $(0, -1)$.

(v) Find the coordinates of the other point on the graph of $f(x)$ at which the tangent to the curve is parallel to T.

SOLUTION

8 (i)

$$f(x)=\frac{1}{x-1}$$

$$f(-2)=\frac{1}{(-2)-1}=\frac{1}{-3}=-\frac{1}{3}=-0.33$$

$$f(0)=\frac{1}{(0)-1}=\frac{1}{-1}=-1$$

$$f(\tfrac{3}{2})=\frac{1}{(\frac{3}{2})-1}=\frac{1}{\frac{1}{2}}=2$$

$$f(5)=\frac{1}{(5)-1}=\frac{1}{4}=0.25$$

8 (ii)

$$f(x)=\frac{1}{(x-1)}=(x-1)^{-1}$$

$$\Rightarrow f'(x)=-1(x-1)^{-2}=-\frac{1}{(x-1)^2}$$

8 (iii)

DRAWING RECIPROCAL GRAPHS

STEPS

1. Find the gap first by putting the bottom of the function equal to zero and solving for x.
2. Find other values of $f(x)$ by putting in values of x as given in the domain.
3. Plot these points by joining up smoothly and continuing towards the vertical line but never touching it.

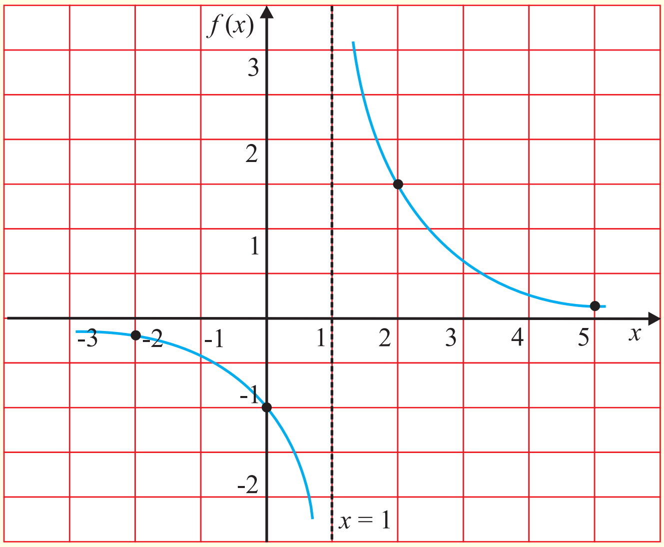
1. Put $(x-1)=0\Rightarrow x=1$

2. Use the values that you worked out in part **(i)**.

Points: $(-2, -\frac{1}{3})$, $(0, -1)$, $(\frac{3}{2}, 2)$, $(5, \frac{1}{4})$

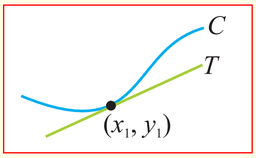
3. Plot the graph. Draw the gap ($x = 1$) first.

CONT....



8 (iv)

STEPS TO FINDING THE EQUATION OF A TANGENT, T, AT A POINT (x_1, y_1), ON THE CURVE, C



STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Substitute x_1 in for x to find the slope of the tangent: $\left(\frac{dy}{dx}\right)_{x=x_1}$
3. Find the point of contact (x_1, y_1) by substituting x_1 into the equation of the curve to find y_1.
4. Find the equation of the line of the tangent using formula **4**.

Equation of a line: $\boxed{y - y_1 = m(x - x_1)}$ **4**

(x_1, y_1) is a point on the line and m is the slope of the line.

1. $\frac{dy}{dx} = f'(x) = -\frac{1}{(x-1)^2}$

2. $\left(\frac{dy}{dx}\right)_{x=0} = -\frac{1}{(0-1)^2} = -\frac{1}{(-1)^2} = -1$

3. The point of contact is given as $(0, -1)$.

4. Equation of the tangent T: Point $(0, -1)$, $m = 1$

$$T : (y - (-1)) = -1(x - 0)$$

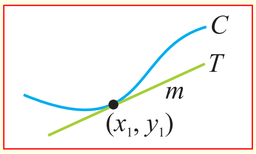
$$\Rightarrow T : y + 1 = -x$$

$$\therefore T : x + y + 1 = 0$$

CONT....

8 (v)

GOING BACKWARDS: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.



STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

A parallel tangent has the same slope as T. $\therefore m = -1$.

1. $\frac{dy}{dx} = -\frac{1}{(x-1)^2}$

2. $\frac{dy}{dx} = -1 \Rightarrow -\frac{1}{(x-1)^2} = -1$

$\Rightarrow 1 = (x-1)^2$

$\Rightarrow \pm 1 = x - 1$

$\therefore x = 0, 2$ [Ignore the first solution as that was used in part **(iv)**.]

3. $y = f(2) = \frac{1}{x-1} = \frac{1}{2-1} = \frac{1}{1} = 1 \Rightarrow (2, 1)$ is the other point.