DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

LESSON NO. 13: CUBIC GRAPHS

2004

6 (b) Let $f(x) = x^3 - 3x^2 + 1,\ x \in \mathbf{R}$.

(i) Find $f(-1)$ and $f(3)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Find the co-ordinates of the local maximum point and of the local minimum point of the curve $y = f(x)$.

(iv) Draw the graph of the function f in the domain $-1 \le x \le 3$.

Use your graph to:

(v) estimate the range of values of x for which $f(x) < 0$ and $x > 0$

(vi) estimate the range of values of x for which $f'(x) < 0$.

SOLUTION

6 (b) (i)

$f(x) = x^3 - 3x^2 + 1$

$\therefore\ f(-1) = (-1)^3 - 3(-1)^2 + 1 = -1 - 3(1) + 1 = -1 - 3 + 1 = -3$

$\therefore\ f(3) = (3)^3 - 3(3)^2 + 1 = 27 - 3(9) + 1 = 27 - 27 + 1 = 1$

6 (b) (ii)

$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$ **1**

REMEMBER IT AS:

Multiply down by the power and subtract one from the power.

CONSTANT RULE: If $y =$ Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$f(x) = x^3 - 3x^2 + 1$

$\Rightarrow\ f'(x) = 3x^2 - 3 \times 2x + 0 = 3x^2 - 6x$

CONT....

6 (b) (iii)

STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.
2. Set $\frac{dy}{dx}=0$ and solve for x to find the turning points.
3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.
4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

1. $y=f(x)=x^3-3x^2+1$

$$\frac{dy}{dx}=f'(x)=3x^2-6x$$

$$\frac{d^2y}{dx^2}=f''(x)=6x-6$$

2. $\frac{dy}{dx}=0\Rightarrow 3x^2-6x=0$

$$\Rightarrow 3x(x-2)=0$$

$$\therefore x=0,\ 2$$

3. $\left(\frac{d^2y}{dx^2}\right)_{x=0}=6(0)-6=-6<0$

$$\left(\frac{d^2y}{dx^2}\right)_{x=2}=6(2)-6=12-6=6>0$$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}<0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}>0$ 7

4. $y=f(0)=(0)^3-3(0)^2+1=1\Rightarrow(0,\ 1)$ is a local maximum.

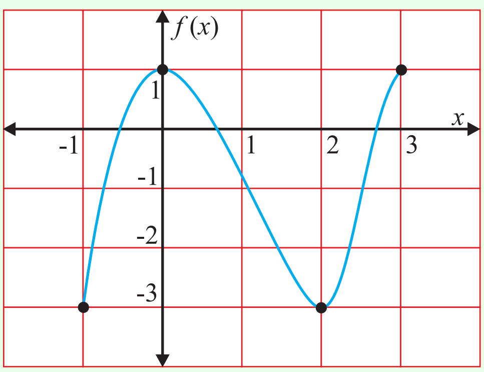
$y=f(2)=(2)^3-3(2)^2+1=8-12+1=-3\Rightarrow(2,\ -3)$ is a local minimum.

6 (b) (iv)

You have enough points already to draw the cubic function.

You have from part **(i)** the starting and finishing points: $(-1,\ -3),\ (3,\ 1)$

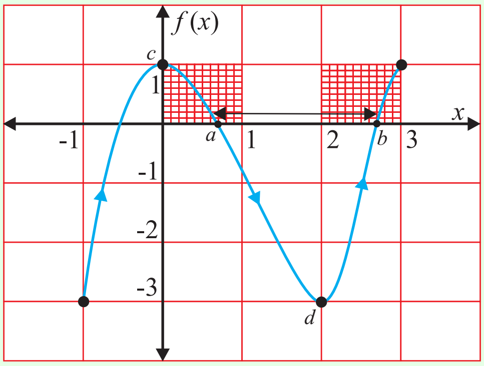
From part **(iii)** you have the turning points: Local maximum $(0,\ 1)$, Local minimum $(2,\ -3)$



CONT....

6 (b) (v)

$f(x)>0$: Above the x-axis; $f(x)<0$: Below the x-axis



What values of x is the graph below the x-axis ($f(x)<0$) and to the right of the y-axis ($x>0$)? You can see from the graph that the values a and b satisfy this condition.

$\therefore 0.7<x<2.7$

6 (b) (vi)

$f'(x)<0$: Curve is decreasing; $f'(x)>0$: Curve is increasing.

You are being asked for what values of x is the curve decreasing. You can see from the graph it decreases as you go from left to right from c to d, i.e. values of x from 0 to 2.

$\therefore 0<x<2$

2001

6 (b) Let $f(x)=2-9x+6x^2-x^3$ for $x\in\mathbf{R}$.

(i) Find $f(-1)$, $f(2)$ and $f(5)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Find the co-ordinates of the local maximum and the local minimum of $f(x)$.

(iv) Draw the graph of $f(x)$ in the domain $-1\le x\le 5$.

(v) Use your graph to find the range of real values of k for which $f(x)=k$ has more than one solution.

Solution

6 (b) (i)

$f(x)=2-9x+6x^2-x^3$

$\Rightarrow f(-1)=2-9(-1)+6(-1)^2-(-1)^3=2+9+6(1)-(-1)$

$=2+9+6+1=18$

$\Rightarrow f(2)=2-9(2)+6(2)^2-(2)^3=2-18+6(4)-(8)$

$=2-18+24-8=0$

$\Rightarrow f(5)=2-9(5)+6(5)^2-(5)^3=2-45+6(25)-(125)$

$=2-45+150-125=-13$

6 (b) (ii)

$y=x^n \Rightarrow \dfrac{dy}{dx}=nx^{n-1}$ **1**

Remember it as:

Multiply down by the power and subtract one from the power.

Constant Rule: If $y=$ Constant $\Rightarrow \dfrac{dy}{dx}=0$

Multiply by a constant rule: If $y=cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx}=c\times\dfrac{du}{dx}$.

$f(x)=2-9x+6x^2-x^3$

$\Rightarrow f'(x)=0-9+6\times 2x-3x^2=-9+12x-3x^2$

6 (b) (iii) **STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:**

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.
2. Set $\frac{dy}{dx}=0$ and solve for x to find the turning points.
3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.
4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

1. $y=f(x)=2-9x+6x^2-x^3$

$$\frac{dy}{dx}=f'(x)=-9+12x-3x^2$$

$$\frac{d^2y}{dx^2}=f''(x)=12-6x$$

2. $\frac{dy}{dx}=0 \Rightarrow -9+12x-3x^2=0$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-1)(x-3)=0$$

$$\therefore x=1,\ 3$$

3. $\left(\frac{d^2y}{dx^2}\right)_{x=1}=12-6(1)=12-6=6$

$$\left(\frac{d^2y}{dx^2}\right)_{x=3}=12-6(3)=12-18=-6$$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP}<0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP}>0$

.... 7

4. $x=1: y=f(1)=2-9(1)+6(1)^2-(1)^3=2-9+6-1=-2 \Rightarrow (1,-2)$ is a local minimum.

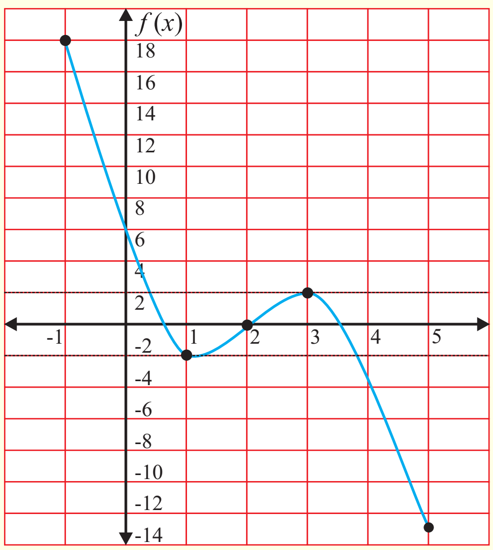
$x=3: y=f(3)=2-9(3)+6(3)^2-(3)^3=2-27+54-27=2 \Rightarrow (3,\ 2)$ is a local maximum.

6 (b) (iv)

You have sufficient information from parts **(i)** and **(iii)** to draw the cubic graph.

Points: $(-1, 18), (2, 0), (5, -13)$

Local maximum: $(3, 2)$; Local minimum: $(1,-2)$



6 (b) (v)

The line $f(x)=k$ is a horizontal line. You can see from the graph that all lines drawn between 2 and -2 will cut the graph in more than one place.

$\therefore -2 \le k \le 2$

1996

8 (b) Let $f(x)=x^3-3x^2$, for $x\in\mathbf{R}$.

(i) Find $f'(x)$, the derivative of $f(x)$. Hence, calculate the coordinates of the local maximum and the local minimum of $f(x)$.

(ii) Draw the graph of

$$f(x)=x^3-3x^2$$

for $-1\le x\le 3$.

SOLUTION

8 (b) (i)

STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.

2. Set $\frac{dy}{dx}=0$ and solve for x to find the turning points.

3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.

4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

1. $y=f(x)=x^3-3x^2$

$$\Rightarrow \frac{dy}{dx}=f'(x)=3x^2-6x$$

$$\Rightarrow \frac{d^2y}{dx^2}=f''(x)=6x-6$$

2. $\frac{dy}{dx}=0\Rightarrow 3x^2-6x=0$

$\Rightarrow 3x(x-2)=0$

$\therefore x=0,\ 2$

3. $\left(\frac{d^2y}{dx^2}\right)_{x=0}=6(0)-6=0-6=-6<0$

$\left(\frac{d^2y}{dx^2}\right)_{x=2}=6(2)-6=12-6=6>0$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}<0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}>0$

.......7

4. $x=0: y=f(0)=(0)^3-3(0)^2=0-0=0\Rightarrow(0,\ 0)$ is a local maximum.

$x=2: y=f(2)=(2)^3-3(2)^2=8-12=-4\Rightarrow(2,\ -4)$ is a local minimum.

CONT....

8 (b) (ii)

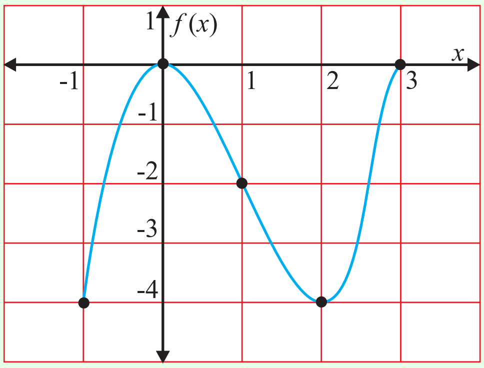
$x=-1: f(-1)=(-1)^3-3(-1)^2=-1-3\times1=-1-3=-4 \Rightarrow (-1,-4)$ is a point.

$x=0: f(0)=(0)^3-3(0)^2=0-3\times0=0-0=0 \Rightarrow (0, 0)$ is a point.

$x=1: f(1)=(1)^3-3(1)^2=1-3\times1=1-3=-2 \Rightarrow (1,-2)$ is a point.

$x=2: f(2)=(2)^3-3(2)^2=8-3\times4=8-12=-4 \Rightarrow (2,-4)$ is a point.

$x=3: f(3)=(3)^3-3(3)^2=27-3\times9=27-27=0 \Rightarrow (3, 0)$ is a point.

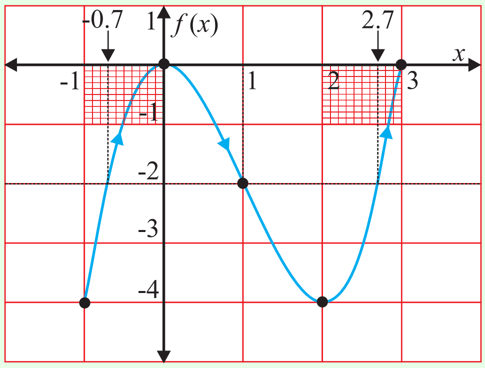


8 (b) (iii)

$f(x)+2=0 \Rightarrow f(x)=-2$

Go to -2 on the $f(x)$ axis. Draw a line straight across until it meets the graph. Read off the x values.

$\therefore x=-0.7, 1, 2.7$



8 (b) (iv)

$f'(x)<0$: Curve is decreasing; $f'(x)>0$: Curve is increasing.

You can see the graph is decreasing from values of x from 0 to 2.

$\therefore f'(x)<0 \Rightarrow 0<x<2$