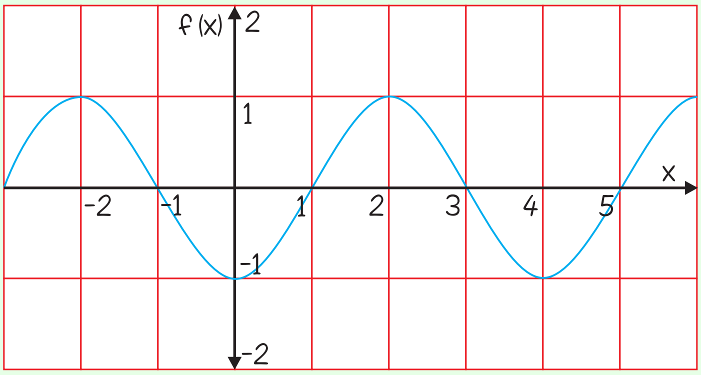
DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2011

6. (a) $f : x \to f(x)$ is a periodic function defined for $x \in \mathbb{R}$.
The period is as indicated in the diagram.



(i) Write down the period and the range of the function.

(ii) Find $f(71)$.

(b) (i) Differentiate $(4x-1)(3-2x^2)$ with respect to x and simplify your answer.

(ii) Given that $y = \dfrac{1}{x^2 - 3x}$, $x \neq 3$, find the range of values of x for which $\dfrac{dy}{dx} < 0$.

(c) Let $f(x) = 2x + \dfrac{1}{x}$, where $x \in \mathbb{R}$ and $x \neq 0$.

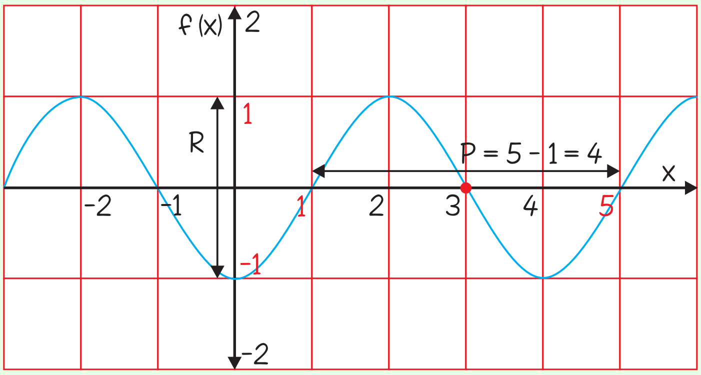
(i) Find the equation of the tangent to the curve $y = f(x)$ at the point $P(1, 3)$.

(ii) Q is another point on the curve $y = f(x)$ such that the tangent at Q is parallel to the tangent at P. Find the co-ordinates of Q.

SOLUTION

6 (a) (i)

1. PERIOD:
The length of the wave along the x-axis before it repeats itself.
2. RANGE:
This is the interval between the lowest y value) and the highest y value.



Period = 4, Range = [–1, 1]

6 (a) (ii)

The value of the function at any value of x can be worked out from the first wave by dividing the value of x by the period and finding the remainder.

$$f(x) = f(\text{Remainder})$$

$$\frac{71}{4} = 18 + \text{Remainder } 3$$

$$f(71) = f(3) = 0$$

6 (b) (i)

$$y = (4x-1)(3-2x^2)$$

$$u = 4x - 1 \quad \Rightarrow \frac{du}{dx} = 4$$

$$v = 3 - 2x^2 \Rightarrow \frac{dv}{dx} = -4x$$

THE PRODUCT RULE: If $y = u \times v$ then: $\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$

$$\frac{dy}{dx} = (4x-1)(-4x) + (3-2x^2)(4)$$

$$= -16x^2 + 4x + 12 - 8x^2$$

$$= -24x^2 + 4x + 12$$

$$= -4(6x^2 - x - 3)$$

6 (b) (ii)

$y=\frac{1}{x^2-3x}=(x^2-3x)^{-1}$ [Bring bracket up and change the sign of power.]

$$\boxed{u=x^2-3x \Rightarrow \frac{du}{dx}=2x-3}$$

$$\boxed{y=(u)^n \Rightarrow \frac{dy}{dx}=n(u)^{n-1}\times\frac{du}{dx}}$$

REMEMBER IT AS:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

$$\frac{dy}{dx}=-1(x^2-3x)^{-2}(2x-3)$$
$$=-\frac{2x-3}{(x^2-3x)^2}$$

$$\frac{dy}{dx}<0 \Rightarrow -\frac{2x-3}{(x^2-3x)^2}<0$$

$$-\frac{(2x-3)\cancel{(x^2-3x)^2}}{\cancel{(x^2-3x)^2}}<0(x^2-3x)^2$$ [Multiply both sides by the denominator.]

$$-(2x-3)<0$$
$$-2x+3<0$$
$$-2x<-3$$
$$x>\frac{-3}{-2}$$
$$x>\tfrac{3}{2}$$

6 (c) (i)

STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Substitute x_1 in for x to find the slope of the tangent: $\left(\frac{dy}{dx}\right)_{x=x_1}$
3. Find the point of contact (x_1, y_1) by substituting x_1 into the equation of the curve to find y_1.
4. Find the equation of the line of the tangent using the equation of a line formula.

$$f(x)=2x+\frac{1}{x}=2x+x^{-1}$$
$$f'(x)=2-1x^{-2}=2-\frac{1}{x^2}$$
$$f'(1)=2-\frac{1}{(1)^2}=2-1=1$$

This is the slope m of the tangent at $x=1$.

Equation of a line: $\boxed{y-y_1=m(x-x_1)}$

$$(x_1,\ y_1)=P(1,\ 3)$$
$$m=1$$
$$y-3=1(x-1)$$
$$y-3=x-1$$
$$0=x-y-1+3$$
$$0=x-y+2$$

6 (c) (ii)

GOING BACKWARDS: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.

STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

$$f'(x) = 2 - \frac{1}{x^2}$$

$$f'(x) = 1 \Rightarrow 2 - \frac{1}{x^2} = 1$$ [You are finding the values of x for which the slope is 1.]

$$2 - 1 = \frac{1}{x^2}$$

$$1 = \frac{1}{x^2}$$

$$1 = x^2$$

$$\pm\sqrt{1} = x$$

$$\pm 1 = x$$ [You already know from part (**i**) that the slope is 1 at $x = 1$. You are interested in the other value of x.]

$$x = -1:\ f(1) = 2(-1) + \frac{1}{(-1)} = -2 - 1 = -3$$

$$\therefore Q(-1, -3)$$

7. **(a)** Differentiate $x^3 - 7x^2 + 6x$ with respect to x.

(b) **(i)** Differentiate $\dfrac{3x+1}{x-2}$ with respect to x.

Write your answer in the form $\dfrac{k}{(x-2)^n}$, where $k, n \in \mathbb{Z}$.

(ii) Given that $y = (x^2 - 2x - 9)^4$, find the value of $\dfrac{dy}{dx}$ when $x = -2$.

(c) A ball is rolled in a straight line along a surface.
The distance, s metres, the ball travels is given by

$$s = 18t - 2t^2$$

where t is the time in seconds from the instant the ball begins to move.

(i) Find the speed of the ball after 3 seconds.

(ii) How far is the ball from the starting point when it stops moving?

(iii) Show that the speed of the ball decreases at a constant rate while it is moving.

SOLUTION

7 (a)

$$y = x^3 - 7x^2 + 6x$$

$$\frac{dy}{dx} = 3x^2 - 14x + 6$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

REMEMBER IT AS:

Multiply down by the power and subtract one from the power.

7 (b) (i)

$$y = \frac{3x+1}{x-2}$$

THE QUOTIENT RULE: If $y = \dfrac{u}{v}$ then: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$u = 3x+1 \Rightarrow \frac{du}{dx} = 3$$

$$v = x-2 \Rightarrow \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{(x-2)(3) - (3x+1)(1)}{(x-2)^2}$$

$$= \frac{3x - 6 - 3x - 1}{(x-2)^2}$$

$$= \frac{-7}{(x-2)^2}$$

7 (b) (ii)

$y=(x^2-2x-9)^4$

$$u=x^2-2x-9 \Rightarrow \frac{du}{dx}=2x-2$$

$$y=(u)^n \Rightarrow \frac{dy}{dx}=n(u)^{n-1}\times\frac{du}{dx}$$

REMEMBER IT AS:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

$$\frac{dy}{dx}=4(x^2-2x-9)^3(2x-2)$$

$$\left(\frac{dy}{dx}\right)_{x=-2}=4((-2)^2-2(-2)-9)^3(2(-2)-2)$$

$$=4(4+4-9)^3(-4-2)$$

$$=4(-1)^3(-6)$$

$$=4(-1)(-6)$$

$$=24$$

7 (c) (i)

$$v=\frac{ds}{dt}$$

$$a=\frac{dv}{dt}$$

SOME NOTES ABOUT THESE TYPES OF PROBLEMS:
Initial distance, speed, acceleration: Put $t = 0$.
When means find t.
Where means find s.
At rest means $v = 0$.

Draw up a s, v, a table as shown on the right.

$$s=18t-2t^2$$
$$v=\frac{ds}{dt}=18-4t$$
$$a=\frac{dv}{dt}=-4$$

$$v=\frac{ds}{dt}=18-4t$$

$$\left(\frac{ds}{dt}\right)_{t=3}=18-4(3)=18-12=6\ \text{m/s}$$

7 (c) (ii)

Work out how long it takes the ball to come to rest by puuting $v = 0$ and solving for t.

$$v=0 \Rightarrow 18-4t=0$$

$$18=4t$$

$$\tfrac{18}{4}=t$$

$$\therefore t=\tfrac{9}{2}\ \text{s}$$

Now work out the distance travelled after this time of 4.5 s.

$$s=18t-2t^2$$

$$=18(\tfrac{9}{2})-2(\tfrac{9}{2})^2$$

$$=9(9)-2(\tfrac{81}{4})$$

$$=81-(\tfrac{81}{2})$$

$$=\tfrac{81}{2}=40.5\ \text{m}$$

7 (c) (iii)

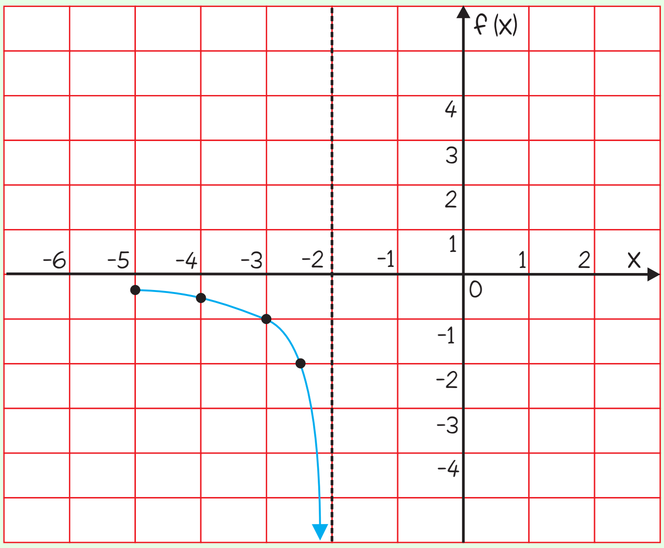
$a=-4\ \text{m/s}^2$ [Acceleration a is the rate of change of speed. A negative value means the speed is decreasing. There is a contant of 4 in the answer (no variable) meaning the rate is constant.]

8. Let $f(x) = \dfrac{1}{x+2}$, where $x \in \mathbb{R}$ and $x \neq -2$.

(i) Copy and complete the following table:

x	-5	-4	-3	-2.5	-1.5	-1	0	1
$f(x)$		-0.5	-1	-2				

(ii) The diagram shows part of the graph of the function f.
Copy and complete the graph from $x = -5$ to $x = 1$.



(iii) On the same diagram, draw the graph of the function $g(x) = x + 2$ in the domain $-5 \leq x \leq 1$, where $x \in \mathbb{R}$.

(iv) Use your graphs to estimate the range of values of x for which $f(x) \leq g(x)$.

(v) Prove that the curve $y = f(x)$ has no turning points.

SOLUTION

8 (i), (ii)

$$f(x)=\frac{1}{x+2}$$

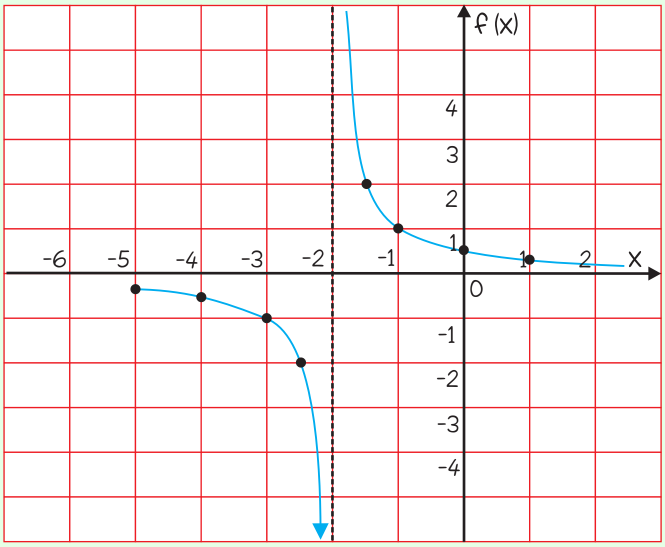
$$f(-5)=\frac{1}{(-5)+2}=-\frac{1}{3}$$

$$f(-1.5)=\frac{1}{(-1.5)+2}=\frac{1}{0.5}=2$$

$$f(-1)=\frac{1}{(-1)+2}=\frac{1}{1}=1$$

$$f(0)=\frac{1}{(0)+2}=\frac{1}{2}$$

$$f(1)=\frac{1}{(1)+2}=\frac{1}{3}$$



x	-5	-4	-3	-2.5	-1.5	-1	0	1
$f(x)$	$-\frac{1}{3}$	-0.5	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$

8 (iii)

$g(x)=x+2$

$g(-5)=(-5)+2=-3$

$g(-4)=(-4)+2=-2$

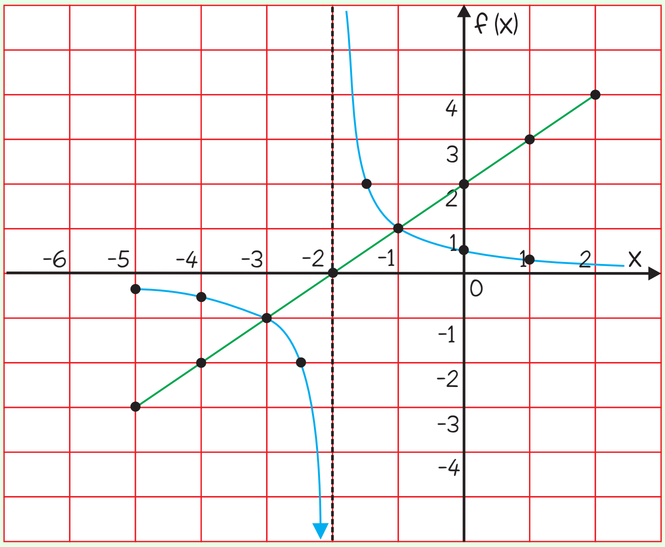
$g(-3)=(-3)+2=-1$

$g(-2)=(-2)+2=0$

$g(-1)=(-1)+2=1$

$g(0)=(0)+2=2$

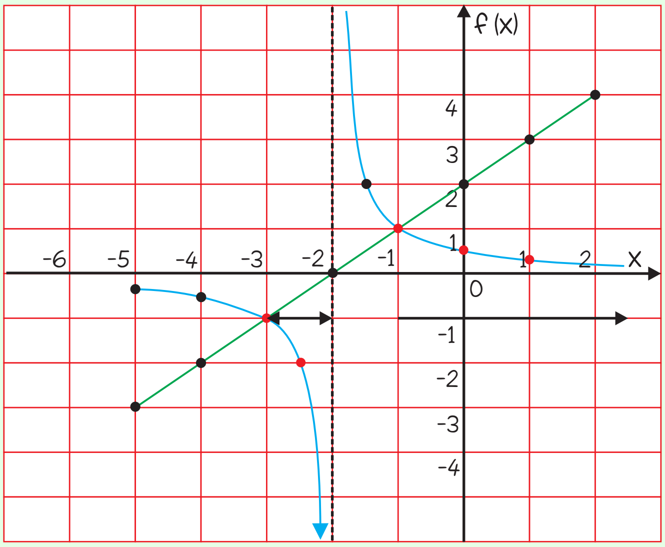
$g(1)=(1)+2=3$



x	-5	-4	-3	-2	-1	0	1	2
$g(x)$	-3	-2	-1	0	1	2	3	4

8 (iv)

You need to find the values of x for which the $f(x)$ is less than or equal to the $g(x)$. In other words, find the values of x for which the blue graph is below or meets the straight line green graph.



$$f(x) \le g(x) : -3 \le x \le -2,\ x \ge -1$$

8 (v)

$$f(x) = \frac{1}{x+2} = (x+2)^{-1}$$

$$f'(x) = -1(x+2)^{-2}(1)$$

$$= -\frac{1}{(x+2)^2}$$

$$y = (u)^n \Rightarrow \frac{dy}{dx} = n(u)^{n-1} \times \frac{du}{dx}$$

REMEMBER IT AS:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

$$f'(x) = 0 \Rightarrow -\frac{1}{(x+2)^2} = 0$$ [Find the turning points by putting the derivative equal to 0.]

$$\Rightarrow -1 = 0$$ [This equation has no solutions for x.]

Therefore, there are no turning points.